LOGIC AND SET THEORY - HOMEWORK 1

OHAD LUTZKY, MAAYAN KESHET

1. Question 1

In the order '$A \in B$', '$A \subseteq B$', '$A \cup B = \emptyset$',

- Yes, no, yes
- No, yes, no
- Yes, no, yes
- Same as 1
- Yes, no, yes
- No, yes, no

2. Question 2

- n
- 0
- $n+1$
- Unknown - either n or $n-1$, depending on whether $\{\emptyset\} \in A$
- 2
- 2
- $2^n + n$
- 2^n

One set with two elements, for which each element is a subset of it, is $\{\emptyset, \{\emptyset\}\}$.

3. Question 3

3.1. Part A.

Proof. We will show mutual containment, from left to right.

$$\begin{aligned} & a \in A \cap (B \cup C) \\ \iff & a \in A \text{ and } a \in B \cup C \\ \iff & a \in A \text{ and } (a \in B \text{ or } a \in C) \\ \iff & (a \in A \text{ and } a \in B) \text{ or } (a \in A \text{ and } a \in C) \\ \iff & a \in (A \cap B) \cup (A \cap C) \end{aligned}$$

□

3.2. Part B.

Proof. We will show mutual containment, from left to right.

$$\begin{aligned} & a \in A \cup (B \cap C) \\ \iff & a \in A \text{ or } a \in B \cap C \\ \iff & a \in A \text{ or } (a \in B \text{ and } a \in C) \\ \iff & (a \in A \text{ or } a \in B) \text{ and } (a \in A \text{ or } a \in C) \\ \iff & a \in (A \cup B) \cap (A \cup C) \end{aligned}$$

□

4. Question 4

4.1. **Part A.** The claim is true.

Proof. We know that $X \subseteq X', Y \subseteq Y'$. This means that, for any x, if $x \in X$ then $x \in X'$, and if $x \in Y$ then $x \in Y'$. Now, if $z \in X + Y$, this means (by the definition of $X + Y$ that $z = x + y$ such that $x \in X$, $y \in Y$. However, as we've shown, that means $x \in X'$ and $y \in Y'$, therefore $z = x + y$ such that $x \in X'$ and $y \in Y'$, which means that $z \in X' + Y'$.

□

4.2. **Part B.** The claim is false.

Take X to be the real numbers and Y to be the imaginary. Take X' to be $X \cup \{\sqrt{-1}\}$ and Y' to be $Y \cup \{1\}$. Obviously, $X \subsetneq X', Y \subsetneq Y'$. But $X + Y = \mathbb{C}$, and $X' + Y' = \mathbb{C}$ as well, so $X + Y = X' + Y'$, and the claim is false[1].

5. Question 5

5.1. **Part A.** The claim is true.

Proof. We will show mutual containment.

$$
\begin{aligned}
& X \in \wp(A) \cap \wp(B) \\
\iff & X \subseteq A \cap B \\
\iff & x \in X \Rightarrow x \in A \text{ and } x \in B \\
\iff & X \subseteq A \text{ and } X \subseteq B \\
\iff & X \in \wp(A) \text{ and } X \in \wp(B) \\
\iff & X \in \wp(A) \cap \wp(B)
\end{aligned}
$$

□

5.2. **Part B.** The claim is true.

Proof. First we'll show WLOG that if $A \subseteq B$, then $\wp(A \cup B) = \wp(A) \cup \wp(B)$.

If $A \subseteq B$, then if $x \in A$ then $x \in B$. Therefore, if $x \in A \cup B$, then either $x \in B$, or $x \in A$ - but as we've shown, this means $x \in B$. Therefore $A \cup B \subseteq B$, and since $B \subseteq A \cup B$, we've shown $A \cup B = B$. Thus what we have left to prove is $\wp(B) = \wp(A) \cup \wp(B)$. Again, $\wp(B) \subseteq \wp(A) \cup \wp(B)$, so we only have to show the reverse containment.

$X \in \wp(A) \Rightarrow X \subseteq A$, which means that if $x \in X$, then $x \in A$. However, we know that $A \subseteq B$, so we have $x \in B$, so we have $X \subseteq B$ and therefore $X \in \wp(B)$. We've shown that $\wp(A) \subseteq \wp(B)$, and as we've seen, this shows that $\wp(A) \cup \wp(B) \subseteq \wp(B)$. All in all, we've shown that $\wp(A \cup B) = \wp(A) \cup \wp(B)$.

Now we will show the other direction - if $\wp(A \cup B) = \wp(A) \cup \wp(B)$, then either $A \subseteq B$ or $B \subseteq A$. Assume by negation that $A \not\subseteq B$ and $B \not\subseteq A$. Therefore there exists $a \in A \setminus B$ and $b \in B \setminus A$. Examine the set $F = \{a, b\}$. $a \in A, b \in B$, therefore $F \subseteq A \cup B$, meaning $F \in \wp(A \cup B)$. Therefore, either $F \in \wp(A)$ or $F \in \wp(B)$, meaning either $F \subseteq A$ or $F \subseteq B$. $F \not\subseteq A$, because $b \in F$ and $b \notin A$, therefore $F \subseteq B$. But $F \not\subseteq B$, because $a \in F$ and $a \notin B$. We have a contradiction to the assumption, and therefore it is false - either $A \subseteq B$, or $B \subseteq A$.

□

[1] If there's anything wrong with that example, replace "real" with "even", "imaginary" with "odd", "1" with 0, "$\sqrt{-1}$" with "1", and "$\mathbb{C}$" with "$\mathbb{Z}$".

5.3. **Part C.** The claim is not true. Take A to be the even numbers and B the odd. No even number is odd or vice versa, therefore $A \setminus B = \emptyset$. For any set G, $\emptyset \subseteq G$, and therefore $\emptyset \in \wp(G)$. Therefore $\emptyset \in \wp(A \setminus B)$, $\emptyset \in \wp(A)$, and $\emptyset \in \wp(B)$. However, this means that $\emptyset \notin \wp(A) \setminus \wp(B)$, and therefore $\wp(A) \setminus \wp(B) \not\subseteq \wp(A \setminus B)$, and the claim is false.

6. Question 6

6.1. **Part A.** The claim is true.

Proof. We will prove that $\bigcup_{i\in\mathbb{N}} \Pi_i \subseteq \bigcup_{i\in\mathbb{N}} \Sigma_i$, and without loss of generality, this will show us the opposite containment as well - and thus we have set equality.

Let us take x such that $x \in \bigcup_{i\in\mathbb{N}} \Pi_i$. This means that there exists an i such that $x \in \Pi_i$. We know that $\Pi_i \subsetneq \Delta_{i+1}$, which tells us that for $j = i + 1$, $x \in \Delta_j$. We also know that $\Delta_i \subsetneq \Sigma_i$, so since $x \in \Delta_j$, we now have $x \in \Sigma_j$. We have shown, therefore, that there exists a j such that $x \in \Sigma_j$, which means that $x \in \bigcup_{i\in\mathbb{N}} \Sigma_i$. □

6.2. **Part B.** Not true. As a counterexample, take $X = \mathbb{R}$. Now we'll define the sets Π, Σ, Δ: $\Sigma_i = \{0, 1, 2, \ldots, 2i\}, \Pi_i = \Sigma_i = \{0, 1, 2, \ldots, 2i, 2i+1\}$. The conditions of the question hold: $\Pi_i = \Sigma_i = \{0, 1, 2, \ldots, 2i, 2i+1\} = \{0, 1, 2, \ldots, 2i\} \cup \{2i+1\} = \Delta_i \cup \{2i + 1\}$, so we have $\Delta_i \subsetneq \Pi_i$ and $\Delta_i \subsetneq \Sigma_i$, and identically - $\Pi_i \subsetneq \Delta_{i+1}$ and $\Sigma_i \subsetneq \Delta_{i+1}$. Now, assume by negation that $\bigcup_{i\in\mathbb{N}} \Delta_i = X$. $\sqrt{2} \in X$ (for our choice $X = \mathbb{R}$), therefore there exists some i for which $\sqrt{2} \in \Delta_i$, which is absurd since we've constructed Δ_i out of natural numbers only. Therefore it cannot be that $\bigcup_{i\in\mathbb{N}} \Delta_i = X$.

LOGIC AND SET THEORY - HW 2

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1. Question 1

1.1. Part A. $\langle a \rangle, b = \{\{a\}, \{a, b\}\}$

1.1.1. *(i).* $\bigcup \langle a \rangle, b = \{a\} \cup \{a, b\} = \{a, b\}$

1.1.2. *(ii).* $\bigcap \langle a \rangle, b = \{a\} \cap \{a, b\} = \{a\}$

1.2. Part B.

1.2.1. *(i).* This implementation meets the demand. First we'll prove that $a = a'$ and then we'll prove that $b = b'$.

Proof. $\{\{a\}, \{a, \{b\}\}\} = \{\{a'\}, \{a', \{b'\}\}\}$. Therefore, $\{a\} \in \{\{a'\}, \{a', \{b'\}\}\}$, which means that either $\{a\} = \{a'\}$ and we're done or that $\{a\} = \{a', \{b'\}\}$, which means either $\{a\} = \{a'\}$ and we're done or $a = \{b'\}$. If $a = \{b'\}$ then $\{\{a\}, \{a, \{b\}\}\} = \{\{a'\}, \{a', a\}\}$ which means that either $\{a\} = \{a'\}$ and we're done or that $\{a\} = \{a', a\}$, which by itself means $\{a\} = \{a'\} \Rightarrow a = a'$. Therefore $a = a'$. Now we'll prove the same for b and b'. $\{\{a\}, \{a, \{b\}\}\} = \{\{a'\}, \{a', \{b'\}\}\}$. Therefore, $\{a, \{b\}\} \in \{\{a'\}, \{a', \{b'\}\}\}$, which means that either $\{a, \{b\}\} = \{a'\}$ or $\{a, \{b\}\} = \{a', \{b'\}\}$.

If $\{a, \{b\}\} = \{a'\}$ then $\{b\} = a' = a$. Therefore, $\{\{a\}, \{a, \{b\}\}\} = \{\{a\}, \{a, a'\}\} = \{\{a'\}, \{a', a'\}\} = \{\{a'\}\}$. Therefore $\{\{a\}, \{a, \{b\}\}\} = \{\{a'\}\} = \{\{a'\}, \{a', \{b'\}\}\} \Rightarrow \{a'\} = \{a', \{b'\}\} \Rightarrow \{b'\} = a' = \{b\} \Rightarrow b = b'$.

If $\{a, \{b\}\} = \{a', \{b'\}\}$ then $\{b\} \in \{a', \{b'\}\}$. Therefore, either $\{b\} = \{b'\}$ and we're done or $\{b\} = a' = a$ and as we have shown before $\{b\} = a' = a \Rightarrow b = b'$. □

1.2.2. *(ii).* This implementation meets the demand.

Proof. Let $\langle a, b \rangle_o = \{\{a\}, \{a, b\}\}$ be the original model we used for order pairs. Therefore, with this model, $\langle a, b \rangle = \{\langle a, b \rangle_o\}$. Obviously, if $a = a', b = b'$, then $\langle a, b \rangle = \langle a', b' \rangle$, so we'll show the other direction.

Assume $\langle a, b \rangle = \langle a', b' \rangle$. Therefore, $\{\langle a, b \rangle_o\} = \{\langle a', b' \rangle_o\}$, which means that $\langle a, b \rangle_o = \langle a', b' \rangle_o$. As proved in class, this means that $a = a', b = b'$.

□

1.2.3. *(iii).* This implementation does not meet the demand. For $a = \{0\}, b = 1, a' = \{1\}, b' = 0$, we have $\langle a, b \rangle = \langle a', b' \rangle = \{\{0\}, \{1\}\}$.

1.3. Part C.

1.3.1. *(i).*

Proof. Note that $\{a, \{b\}\} \subseteq \wp(B) \cup A$. This shows that $\{\{a, \{b\}\} \subseteq \wp(\wp(B) \cup A)$, which in turn shows that $\{\{a\}, \{a, \{b\}\}\} \subseteq \wp(A \cup \wp(B)) \cup \wp(A)$. Therefore,

$$A \times B = \left\{ \{\{a\}, \{a, \{b\}\}\} \in \wp(\wp(A) \cup \wp(A \cup \wp(B))) | a \in A, b \in B \right\}$$

□

1.3.2. *(ii).*

Proof. If we define $\times_o$ to be a cartesian product of two sets using $\langle , \rangle_o$, then we've shown in class that for any two sets A, B, there exists a set $X = A \times_o B$. By the base assumption of the existence of the powerset of each set, we know there exists $\wp(X)$. For our current ordered pair model, $\langle a, b \rangle = \{\langle a, b \rangle_o\} \in \wp(X)$, therefore the following set exists:

$$A \times B = \left\{ \{\langle a, b \rangle_o\} \in \wp(X) | a \in A, b \in B \right\}$$

□

2. Question 2

2.1. **Part A.** The set does not exist.

Proof. Let P be the universal set of powersets, $\mathcal{P} = \{\wp(A) : A \text{ is a set}\}$. Let

$$P_0 = \{\wp(X) \in \mathcal{P} : \wp(X) \notin X\}$$

Assume $\wp(P_0) \in P_0$. Therefore, by definition of P_0, $\wp(P_0) \notin P_0$. Therefore, again by definition of P_0, $\wp(P_0) \in P_0$. We have a contradiction, therefore $\mathcal{P}$ cannot exist.

□

2.2. **Part B.** This set does not exist.

Proof. Let $\mathcal{R} = \{R \subseteq A \times B : A, B \text{ are sets}\}$ be the set of all relations. Therefore exists the set $\bigcup \mathcal{R}$, which - since A, B can be any sets, and we join all subsets, is $U \times U$, U being the universal set. But then exists $\text{dom}(U \times U) = U$, which we have proven not to exist.

□

3. Question 3

3.1. **Part A.** This part is true.

Proof. $x \in R^{-1}(B_1 \cup B_2)$. This is true iff exists $y \in B_1 \cup B_2$ such that $(x, y) \in R$, which in turn is true iff exists such a y either in B_1 or B_2. This is true iff $x \in R^{-1}(B_1)$ or $x \in R^{-1}(B_2)$, or in other words, $x \in R^{-1}(B_1) \cup R^{-1}(B_2)$.

□

3.2. **Part B.** This part is false. Assume $A = \{0\}, B = \{0, 1\}, R = \{\langle 0, 0 \rangle, \langle 0, 1 \rangle\}$, and take $B_1 = \{0\}, B_2 = \{1\}$. Therefore, $R^{-1}(B_1) = R^{-1}(B_2) = \{0\}$, however $R^{-1}(B_1 \cap B_2) = R^{-1}(\emptyset) = \emptyset$.

4. Question 4

4.1. **Part A.** This part is false. Take $A = \{1, 2\}, R_1 = \{\langle 1, 1 \rangle, \langle 1, 2 \rangle, \langle 1, 3 \rangle\}$ and $R_2 = \{\langle 2, 2 \rangle, \langle 2, 1 \rangle, \langle 1, 1 \rangle\}$. It's easy to see that $R_1 \cup R_2$ isn't antisymmetric.

4.2. **Part B.** This part is true.

Proof. Assuming R_1, R_2 are partial orders over A, we will show that $R_1 \cap R_2$ is a partial order.

Reflexivity: R_1 is a P.O. over A^2, therefore it is reflexive, and $a \in A \Rightarrow \langle a, a\rangle \in R_1$. Similarily, R_2 is a P.O. over A^2, thus $a \in A \Rightarrow \langle a, a\rangle \in R_2$. So we have $a \in A \Rightarrow \langle a, a\rangle \in R_1 \cap R_2$.

Antisymmetry: If $\langle x, y\rangle, \langle y, x\rangle \in R_1 \cap R_2$, then $\langle x, y\rangle, \langle y, x\rangle \in R_1$, therefore since R_1 is antisymmetric, $x = y$

Transitivity: If $\langle x, y\rangle, \langle y, z\rangle \in R_1 \cap R_2$, then $\langle x, y\rangle, \langle y, z\rangle \in R_1$, so by transitivity of R_1, $\langle x, z\rangle \in R_1$, and $\langle x, y\rangle, \langle y, z\rangle \in R_2$, so similarily $\langle x, z\rangle \in R_2$, therefore $\langle x, z\rangle \in R_1 \cap R_2$.

□

4.3. **Part C.** This part is true.

Proof. Assume by negation $R_1 \neq R_2$. $R_1 \subseteq R_2$, therefore $R_1 \subsetneq R_2$. Therefore exists $R_2 \ni \langle a, b\rangle \notin R_1$. $\langle a, b\rangle \in R_2$, therefore $a, b \in A$. R_1 is a F.O. over A, therefore either $\langle a, b\rangle$ or $\langle b, a\rangle \in R_1$, and we've already ruled out $\langle a, b\rangle$, so $\langle b, a\rangle \in R_1$. However, $R_1 \subseteq R_2$, therefore $\langle b, a\rangle \in R_2$, and since also $\langle a, b\rangle \in R_2$, we have $a = b$, by antisymmetry of R_2. Therefore, by reflexivity of R_1, $\langle a, b\rangle \in R_1$, in contradiction to the assumption. Therefore $R_1 = R_2$. □

5. Question 5

5.1. **Part A.** This claim is true.

Proof. Assume by negation $m, n \in A, m \neq n$ are both a minimum element in A. Because m is a minimum element, by defenition $(m, n) \in R$. Similarly, because n is a minimum element, by defenition $(n, m) \in R \Rightarrow$contradiction, because R is antisymmetric. Therefore $m = n$. □

5.2. **Part B.** This part is false. Take $A = \mathbb{Z} \cup \{0.5\}$ and $R = \{(a, b) \in Z^2 : a \leq b\} \cup \{(0.5, 0.5)\}$. 0.5 is uniquely minimal, but not a minimum - $(0, 0.5) \notin R$.

5.3. **Part C.**

Proof. We'll prove by induction on $|A|$. For $|A| = 1$, a being the single element of the set, the only possible relation is $\langle a, a\rangle$, therefore a is minimal, and we're done.

Now, assuming the claim is true for $|A| = n$, we'll prove for $|A| = n + 1$. We know A is finite, therefore there is a 1-1 function from A on $\{1, \ldots, n\}$, n being $|A|$. Let a_i be the inverse of one such function (it is 1-1 and on, so it has an inverse function). Let $A' = A \setminus a_1, R' = R \setminus \{\langle x, y\rangle | x = a_1 \text{ or } y = a_1\}$. $|A'|$ would be n, therefore there is a minimal element a_k of A' by R', and $k \neq 1$ (because a_1 isn't in A'). Now we will check minimality for a_1 and a_k by looking at all possible options:

- If neither $\langle a_1, a_k\rangle$ nor $\langle a_k, a_1\rangle$ are in R, then a_k is minimal (and so is a_1), so we're done.
- If $\langle a_k, a_1\rangle \in R$, then by antisymmetry $\langle a_1, a_k\rangle \notin R$, and thus a_k is minimal.
- If $\langle a_1, a_k\rangle \in R$, we'll show a_1 is minimal: Assume by negation it is not, therefore there exists $A \ni a_j \neq a_1, a_k$ such that $\langle a_j, a_1\rangle \in R$. By transitivity of R, $\langle a_j, a_k\rangle \in R$, and by definition of R', $\langle a_j, a_k\rangle \in R'$, in contradiction with a_k being minimal in A' by R'.

□

6. Question 6

6.1. Part A. The claim is true.

Proof. R is an equivelance, we'll show that it is a sharing relation. Assume that $\langle a, b\rangle, \langle a, c\rangle \in R$. By symmetry, $\langle b, a\rangle \in R$ as well, and by transitivity, $\langle b, c\rangle \in R$. □

6.2. Part B. The claim is true.

Proof. Reflexivity we already have, so we'll show symmetry and transitivity.

- **Symmetry:** Assume $a, b \in A, \langle a, b\rangle \in R$. Because of reflexivity, we have that $\langle a, a\rangle, \langle b, b\rangle \in R$. Since $\langle a, b\rangle, \langle a, a\rangle \in R$, by sharing we have that $\langle b, a\rangle \in R$.
- **Transitivity:** Assume $a, b, c \in A, \langle a, b\rangle, \langle b, c\rangle \in R$. By reflexivity we have that $\langle a, a\rangle, \langle b, b\rangle, \langle c, c\rangle \in R$, and by symmetry (we've proven), we have that $\langle b, a\rangle \in R$. Therefore, by sharing we have that $\langle a, c\rangle \in R$.

□

6.3. Part C. The claim is false.

$\{\langle 1, 3\rangle, \langle 1, 2\rangle, \langle 2, 3\rangle\}$ is a sharing relation, but it is not symmetric.

LOGIC AND SET THEORY HW 3

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1. Question 2

1.1. Part A. We need to prove that $L = \left\{(a,a) \in A^2 | a \in \text{range}(R)\right\} \subseteq R^{-1} \circ R$

Proof. $(a,a) \in L$, therefore $a \in \text{range}(R)$. Therefore exists b such that $(b,a) \in R$, which means $(a,b) \in R^{-1}$. We've shown that there exists a "shared" b such that $(a,b) \in R^{-1}, (b,a) \in R$, therefore $(a,a) \in R^{-1} \circ R$. □

1.2. Part B. We need to prove that $L' = \left\{(a,a) \in A^2 | a \in \text{dom}(R)\right\} \subseteq R \circ R^{-1}$.

Proof. $(a,a) \in L'$, therefore $a \in \text{dom}(R)$. Therefore exists b such that $(a,b) \in R$, which means $(b,a) \in R^{-1}$. Therefore, as before, $(a,a) \in R \circ R^{-1}$. □

1.3. Part C. The assumption that for each $a \in A$ there is at most one b so $(a,b) \in R$ can be expressed thus: If $(a,b), (a,b') \in R$, then $b = b'$. Now we want to show equality - we've shown one direction in (**??**), so we we'll show the other - that is, that $R^{-1} \circ R \subseteq L$.

Proof. $(a,b) \in R^{-1} \circ R$. Therefore there exists c such that $(a,c) \in R^{-1}, (c,b) \in R$. We then know that $(c,a) \in R$, and since also $(c,b) \in R$, then by the assumption, $a = b$. Furthermore, $(c,a) \in R$, which means that $a \in \text{range}(R)$, and thus $(a,b) \in L$. □

1.4. Part D. Assume that if $(a,b), (a',b) \in R$ then $a = a'$. Then the claim is true. Again, we only have to show that $R \circ R^{-1} \subseteq L'$.

Proof. $(a,b) \in R \circ R^{-1}$, therefore exists c so $(a,c) \in R, (c,b) \in R^{-1}$. Then $(b,c) \in R$, and by our assumption, we have $a = b$. Furthermore, $(a,c) \in R$, which means that $a \in \text{dom}(R)$, and altogether we have $(a,b) \in L'$. □

2. Question 3

2.1. Part A. No, take $R = \{(1,1),(1,2),(2,1),(2,2)\}$ and $S = \{(2,2),(2,3),(3,2),(3,3)\}$. R and S are equivalences, but $(1,2),(2,3) \in R \cup S \nRightarrow (1,3) \notin A \cup B$.

2.2. Part B. Yes, $R \cap S$ is an equivalence.
Proof.

- **Reflexivity:** $a \in A$ and R, S are equivalences $\Rightarrow (a,a) \in R, S \Rightarrow (a,a) \in R \cap S$.
- **Symmetry:** $(a,b) \in R \cap S \Rightarrow (a,b) \in R, S \Rightarrow (b,a) \in R, S \Rightarrow (b,a) \in R \cap S$.
- **Transitivity:** $(a,b),(b,c) \in R \cap S \Rightarrow (a,b),(b,c) \in R, S \Rightarrow (a,c) \in R, S \Rightarrow (a,c) \in R \cap S$.

□

2.3. **Part C.** Yes, R^{-1} is an equivalence.

Proof.

Reflexivity: $a \in A \Rightarrow (a,a) \in R \Rightarrow (a,a) \in R^{-1}$.

Symmetry: $(a,b) \in R^{-1} \Rightarrow (b,a) \in R$ and by symmetry of R, $(a,b) \in R \Rightarrow (b,a) \in R^{-1}$.

Transitivity: $(a,b),(b,c) \in R^{-1} \Rightarrow (b,a),(c,b) \in R$ and by symmetry of R, $(a,b),(b,c) \in R$ and because of transitivity of R, $(a,c) \in R$ and by symmetry of R, $(c,a) \in R \Rightarrow (a,c) \in R^{-1}$.

□

2.4. **Part D.** The claim is false. Take $R = \{(1,1),(2,2),(3,3),(1,2),(2,1)\}$, $S^{-1} = \{(1,1),(2,2),(3,3),(2,3),(3,2)\}$, therefore

$$R \circ S^{-1} = \{(1,1),(2,2),(3,3),(1,2),(1,3),(2,1),(3,2),(3,3)\}$$

and $(3,1) \notin R \circ S^{-1}$

2.5. **Part E.** The claim is true.

Proof. First we'll prove that if $R \neq S$, then $A/R \neq A/S$.

We know that $R \neq S$, so we'll assume WLOG that there is a pair $a,b \in A$ such that $(a,b) \in R \setminus S$. Therefore, $b \in [a]_R$, $b \notin [a]_S$. By definition, $[a]_R \in A/R$, and we'll show that $[a]_R \notin A/S$.

Assume by negation that in fact $[a]_R \in A/S$. We know $[a]_S \in A/S$, and we know[1] that A/S is a division. Since $a \in [a]_R, [a]_S$, then $[a]_R \cap [a]_S \neq \emptyset$, and by definition of a division this is only possible if $[a]_R = [a]_S$. And since $b \in [a]_R$, we have that $b \in [a]_S$, and therefore $(a,b) \in S$, in contradiction to the assumption. Therefore, $A/R \neq A/S$.

Now we'll prove that if $A/R \neq A/S$, then $R \neq S$. We'll assume WLOG that there exists $a \in A$ such that $[a]_R \in A/R$ but $[a]_R \notin A/S$. By definition of A/S, $[a]_S \in A/S$, and since $[a]_R \notin A/S$ this means that $[a]_R \neq [a]_S$. Then, again WLOG, we'll assume that there exists $b \in [a]_R \setminus [a]_S$, and therefore $(a,b) \in R \setminus S$.

□

3. Question 4

3.1. **Part C.**

Proof. First we'll show that $E_{A/R} \subseteq R$: Assume $(a,b) \in E_{A/R}$. Therefore exists a set $p \in A/R$ such that $a,b \in p$. p could be written as $[a]_R$, and we have that $b \in [a]_R$, therefore $(a,b) \in R$.

Now we'll show that $R \subseteq E_{A/R}$. Assume $(a,b) \in R$, therefore exists $[a]_R \in A/R$, and $b \in [a]_R$. Assign $p = [a]_R$, and you have that there exists p such that $a,b \in p$ and $p \in A/R$, therefore $(a,b) \in E_{A/R}$.

□

3.2. **Part D.**

Lemma 1. *Assume P is a division of A, $B \in P$, and $a \in B$. Then $B = [a]_{E_P}$.*

Proof of Lemma **??**. Assume $b \in B$. Then by definition of E_P, $(a,b) \in E_P$, and therefore $b \in [a]_{E_P}$. We've shown $B \subseteq [a]_{E_P}$.

Now assume $c \in [a]_{E_P}$. This means that $(a,c) \in E_P$, and since P is a division over A, then $E_P \subseteq A \times A$, and therefore $c \in A$. Now, by definition of E_P, this means that there is $B' \in P$ such that $a,c \in P$, and since $a \in B$, then $B' \cap B \neq \emptyset$.

[1]See question 4A

And by definition of a division, this means that $B = B'$. Therefore, $c \in B$. We've shown $[a]_{E_P} \subseteq B$.

We have thus shown that $B = [a]_{E_P}$. □

Proof of **??**. Assume $B \in P$, and $a \in B$, then by Lemma **??**, $B = [a]_{E_P}$. Therefore $B \in A/E_P$. We've shown $P \subseteq A/E_P$.

Assume $a \in A$, therefore $[a]_{E_P} \in A/E_P$. By definition of a division, we know that $\bigcup P = A$, therefore there exists $B \in P$ such that $a \in B$. By Lemma **??**, $B = [a]_{E_P}$, and thus $[a]_{E_P} \in P$. We've shown that $A/E_P \subseteq P$.

We have thus shown that $P = A/E_P$. □

4. Question 7

4.1. **Part A.** The claim is false. Take $A = \{0\}, B = \{0, 1\}, F = \{f_1 : x \mapsto 0, f_2 : x \mapsto 1\}$. f_1, f_2 are not onto B, and yet F covers B.

4.2. **Part B.** The claim is true.

Proof. Let $\tilde{f}$ be onto B. Therefore, for each $b \in B$, there is $a \in A$ such that $\tilde{f}(a) = b$, therefore F covers B.

□

4.3. **Part C.** The claim is false. Take $A = \{0, 1\}, B = \{0\}, F = \{f_1 : x \mapsto 0\}$. $C_0 = \{f_1\}$, but $f_1(0) = f_1(1)$, so f_1 isn't 1-1.

4.4. **Part D.** The claim is false. Take $A = B = \{0, 1\}, F = \{f_1 : x \mapsto x, f_2 : x \mapsto 1 - x\}$. f_1, f_2 are 1-1, but $|C_0| = 2$.

5. Question 8

5.1. **Part A.** The claim is false. Take $i = 1, j = 3, f_2 : x \mapsto 2, f_3 : x \mapsto 3$. Obviously, $(f_2, f_3) \in R_3$. Assume by negation that $f(2, f_3) \in R_1 \circ R_3$, then exists z such that $(f_2, z) \in R_1$, therefore $f_2 \in N_1^{\mathbb{N}}$, which it clearly isn't.

5.2. **Part B.** The claim is true.

Proof. We will begin by making a simplification of the definition of R_i. By definition,

$$N_i^{\mathbb{N}} = \left\{ f \in \mathbb{N}^{\mathbb{N}} | \text{For all } k \in \mathbb{N}, f(k) \leq i \right\}$$

Therefore,

$$R_i = \left\{ (f, g) \in \mathbb{N}^{\mathbb{N}} \times \mathbb{N}^{\mathbb{N}} | \text{For all } k \in \mathbb{N}, f(k) \leq g(k) \leq i \right\}$$

Now we will show that $R_j \circ R_i \subseteq R_i$. Assume $(f, g) \in R_j \circ R_i$, therefore there exists z such that $(f, z) \in R_j, (z, g) \in R_i$. This means that for any $k \in \mathbb{N}$, $f(k) \leq z(k) \leq j$ and $z(k) \leq g(k) \leq i$. By transitivity of the $\leq$ relation, we have that $f(k) \leq g(k) \leq i$, therefore $(f, g) \in R_i$.

Now we will show that $R_i \subseteq R_j \circ R_i$. Assume $(f, g) \in R_i$, therefore for all $k \in \mathbb{N}$, $f(k) \leq g(k) \leq i$. Especially, $f(k) \leq f(k) \leq i$, and since $i \leq j$, $f(k) \leq f(k) \leq j$, and therefore $(f, f) \in R_j$. Since $(f, g) \in R_i$ as well, we have that $(f, g) \in R_i \circ R_j$. □

LOGIC AND SET THEORY - HW 4

OHAD LUTZKY, MAAYAN KESHET

1. Question 1

$$B = \{0\}, F = \{x \mapsto x + 2\}$$

2. Question 3

2.1. Part A.
This claim is true.

Proof. Mark $a_1, a_2, \dots, a_{n+k} = \sigma_1, \dots, \sigma_n, \tau_1, \dots, \tau_k$. $\sigma_1, \dots, \sigma_n$ is a creation sequence, therefore for any $1 \leq i \leq n$, either $\sigma_i \in B$ or $\sigma_i = f(\sigma_k, \sigma_l, \sigma_m, \dots)$ such that $f \in F$ and $k, l, m, \dots < i$. Therefore, for any such i, either $a_i \in B$ or $a_i = f(a_k, a_l, a_m, \dots)$ such that $f \in F$ and $k, l, m, \dots < i$. Similarily, $\tau_1, \dots, \tau_k$ is a creation sequence, so for all $n + 1 \leq i \leq n + k$, either $a_i \in B$ or $a_i = f(a_k, a_l, a_m, \dots)$ such that $n + 1 \leq k, l, m, \dots \leq i$, and privately $k, l, m, \dots < i$. Therefore $a_1, \dots, a_{n+k}$ is a creation sequence.

□

2.2. Part B.
This claim is true, and the previous proof holds with a slight change - replace all occurences of n with 2.

2.3. Part C.
This claim is true, and the previous proof holds with alterations. Despite the intertwining of the series, the claim that each a_i is still either an element of B or a function of previous elements holds.

2.4. Part D.
This claim is false. Take $B = \{0\}, F = \{x \mapsto x + 1\}, n = 1, \sigma_1 = 0, k = 3, \tau_1 = 0, \tau_2 = 1, \tau_3 = 2$. Then the proposed sequence is $0, 2, 1, 0$ and the second entry, 2, is not in the base and not a function of 0.

3. Question 4

3.1. Part A.
The claim is false. Let $Y = \mathbb{N}, B = \{\{n\} \in \wp(\mathbb{N}) | n \in \mathbb{N}\}$. We will show that $\bigcup B = \mathbb{N}$ and $\mathbb{N} \notin X_{B,F}$.

Proof. First we will show that $\bigcup B = \mathbb{N}$. $\bigcup B \subseteq \mathbb{N}$: By definition of B, if $n \in A$ and $A \in B$, then $A = \{n\}$ and $n \in \mathbb{N}$. So we will show that $\mathbb{N} \subseteq \bigcup B$. If $n \in \mathbb{N}$, then $\{n\} \in \wp(\mathbb{N})$, and again by definition of B, $\{n\} \in B$, therefore $n \in \bigcup B$. We have shown that $\bigcup B = \mathbb{N}$.

Now we will show that $\mathbb{N} \notin X_{B,F}$. We will do this by showing that for any $A \in X_{B,F}$, A is finite. For the base, this is shown by definition, because each element $b \in B = \{n\}$, and is therefore finite. As for F, we have shown in class that for any two finite sets a, b, $a \cup b$ and $a \cap b$ are finite. Therefore any $A \in X_{B,F}$ is finite. Seeing as $\mathbb{N}$ is not finite, then $\mathbb{N} \notin X_{B,F}$.

□

3.2. **Part B.** The claim is false. Select $Y = \mathbb{N}, B = \{\mathbb{N} \setminus \{n\} \in \wp(\mathbb{N}) | n \in \mathbb{N}\}$. We will show that $\bigcap B \notin X_{B,F}$.

Claim 1. $\bigcap B = \emptyset$

Proof of Claim **??**. Assume by negation that there exists $b \in \bigcap B$. Therefore $b \in \mathbb{N}$ and for any $A \in B$, $b \in A$. But by definition of B, $\mathbb{N} \setminus \{b\} \in B$, therefore $b \notin \bigcap B$. □

Lemma 1. *Assume $C \subseteq \mathbb{N}$ is a finite set, then $\mathbb{N} \setminus C$ is infinite.*

Proof of Lemma **??**. We have shown in class that for any finite set $C \subseteq \mathbb{N}$, there is a maximal element $\max C$. Define $f : \mathbb{N} \to \mathbb{N} \setminus C$ such that $f(i) = \max(C) + 1 + i$. Obviously, $\max(C) + 1 + i \in \mathbb{N} \setminus C$.

We will now show f is 1-1. Assume there exist $i_1, i_2 \in \mathbb{N}$ such that $f(i_1) = f(i_2)$. Then $\max(C)+1+i_1 = \max(C)+1+i_2$, and we have that $i_1 = i_2$. We have shown a 1-1 function from $\mathbb{N}$ to $\mathbb{N} \setminus C$, therefore $N \setminus C$ is infinite. □

Claim 2. *Assume $B = \{\mathbb{N} \setminus \{n\} \in \wp(\mathbb{N}) | n \in N\}, F = \{f_\cap, f_\cup\}$ and let $K = \{\mathbb{N} \setminus C \in \wp(\mathbb{N}) | C \subseteq \mathbb{N}$ is finite $\}$, then $X_{B,F} \subseteq K$.*

Proof of Claim **??**.

Base: Each $A \in B$ is explicitly defined as $\mathbb{N} \setminus \{n\}$, $\{n\}$ obviously being finite. Therefore $B \subseteq K$.

Closure: Assume $A_1, A_2 \in K$. Then by definition, $A_1 = \mathbb{N} \setminus C_1, A_2 = \mathbb{N} \setminus C_2$, and C_1, C_2 are finite. Therefore:

- $f_\cup$**:** By De-Morgan's laws, $f_\cup(A_1, A_2) = (\mathbb{N} \setminus C_1) \cup (\mathbb{N} \setminus C_2 = \mathbb{N} \setminus (C_1 \cap C_2)$, and as we've shown in class that, seeing as C_1, C_2 are finite, so is $C_1 \cap C_2$.
- $f_\cap$**:** By De-Morgan's laws, $f_\cap(A_1, A_2) = (\mathbb{N} \setminus C_1) \cap (\mathbb{N} \setminus C_2 = \mathbb{N} \setminus (C_1 \cup C_2)$, and as we've shown in class that, seeing as C_1, C_2 are finite, so is $C_1 \cup C_2$.

□

Proof of Part **??**. We've shown that $\bigcap B = \emptyset$, therefore $\bigcap B$ is finite. Therefore, by Lemma **??**, cannot be written as $\mathbb{N} \setminus C$, C being finite, therefore $\bigcap B \notin K$. And by Claim **??**, $X_{B,F} \subseteq K$, therefore $\bigcap B \notin X_{B,F}$. □

4. Question 6

Proof. Let $B_v = \{v\}, F = \{f_{\sigma_i} \in \Sigma^* \times \Sigma^* | \sigma_i \in \Sigma, f_{\sigma_i}(w) = w\sigma_i\}$. Then by definition, $Cone(v) = X_{B_v,F}$. We'll also mark $K_v = \{w \in \Sigma^* |$Exists $u \in \Sigma^*$ such that $w = vu\}$. We now need to show that $Cone(v) = K_v$.

We'll show that $K_v \subseteq Cone(v)$. Assume $w \in K_v$, then by definition there exists a word $u \in \Sigma^*$ such that $w = vu$. $u \in \Sigma^*$, so it can be written $u = \sigma_1\sigma_2 \ldots \sigma_n, \sigma_i \in \Sigma$. We will show a creation sequence for vu in $X_{B_v,F}$:

$$\begin{aligned} a_1 &: v && \text{Base} \\ a_2 &: v\sigma_1 && f_{\sigma_1}(a_1) \\ a_3 &: v\sigma_1\sigma_2 && f_{\sigma_2}(a_2) \\ &\vdots \\ a_n &: v\sigma_1\sigma_2 \ldots \sigma_n && f_{\sigma_n}(a_{n-1}) \end{aligned}$$

Therefore $vu \in X_{B_v,F}$, which means $w \in Cone(v)$. We have shown that $K_v \subseteq Cone(v)$.

We will now show that $Cone(v) \subseteq K_v$ by induction.

Base: $v = v\epsilon, \epsilon \in \Sigma^*$[1], therefore $v \in K_v$.

Closure: $w \in K_v$, therefore $w = vu$ for some $u \in \Sigma^*$. For any $\sigma_i \in \Sigma$, $f_{\sigma_i}(w) = vu\sigma_i$. By definition of Σ^*, $u\sigma_i \in \Sigma^*$, therefore $vu\sigma_i = f_{\sigma_i}(w) \in K_v$.

We have shown that $Cone(v) = K_v$.

□

5. Question 7

5.1. Part A.

The claim is true. We will show a creation sequence for $[-7, \infty)$ in $I_{A,P}$.

$$\begin{aligned} a_1 &: [-7, 0] && \text{Base} \\ a_2 &: [0, \infty) && \text{Base} \\ a_3 &: [-7, \infty) && f(a_1, a_2) \end{aligned}$$

5.2. Part B.

The claim is false.

Proof. Let $Y = \{[a, b] \in \wp(\mathbb{R}) | a, b \in \mathbb{Q}, a \leq b\} \cup \{[a, \infty) \in \wp(\mathbb{R}) | a \in \mathbb{Q}, a \leq 0\}$. We will show that $I_{A,P} \subseteq Y$ by induction. Obviously, $[7, \infty) \notin Y$, therefore $[7, \infty) \notin I_{A,P}$.

Base: If $Z = [a, b] \in A$, therefore $Z \in Y$ (we defined the compact segments identically). If $Z = [0, \infty)$, then since $0 \leq 0$, $Z \in Y$ again.

Closure: Assume $Z_1, Z_2 \in Y$. We will show that $f(Z_1, Z_2) \in Y$.

- If $Z_1 = [a, \infty), Z_2 = [b, c]$ or $Z_2 = [b, \infty)$, then since $b \in \mathbb{Q}$, $b \neq \infty$, and thus $f(Z_1, Z_2) = Z_1 \in Y$.
- If $Z_1 = [a, b]$,
  - If $Z_2 = [c, d]$ or $[c, \infty)$, and $c \neq b$, then $f(Z_1, Z_2) = Z_1 \in Y$.
  - If $Z_2 = [b, c]$ then $f(Z_1, Z_2) = [a, c] \in Y$.
  - If $Z_2 = [b, \infty)$ then since $Z_2 \in Y$, $b \leq 0$, and since $Z_1 \in Y$, $a \leq b$, and therefore $a \leq 0$. $f(Z_1, Z_2) = [a, \infty)$, and since $a \leq 0$, we have $f(Z_1, Z_2) \in Y$.

□

5.3. Part C.

Reflextivity: True.

Proof. Take $a \in A$. $a \subseteq a$ and $\min(a) = \min(a)$. Therefore, a is a prefix of $a \Rightarrow (a, a) \in S$.

□

Symmetry: False. Take $a = [4, 5], b = [1, 5]. a = [4, 5] \subseteq [1, 5] = b$ and $\max(a) = 5 = \max(b)$. Therefore, a is a suffix of $b \Rightarrow (a, b) \in S$. But, $b = [1, 5] \nsubseteq [4, 5] = a \Rightarrow b$ is neither a prefix nor a suffix of a. $\Rightarrow (b, a) \notin S$.

Anti-Symmetry: True.

Proof. Assume $(a, b), (b, a) \in S$. We'll show $a = b$. $(a, b) \in S \Rightarrow a \subseteq b$ and $(b, a) \in S \Rightarrow b \subseteq a$. Therefore, $a = b$.

□

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[1] It was not explicitly specified that the empty work $\epsilon \in \Sigma^*$, but the claim is false otherwise

Transitivity: False. Take $a = [2,3], b = [1,3], c = [1,4] \in A$. $a = [2,3] \subseteq [1,3] = b$ and $\max(a) = 3 = \max(b)$. Therefore, a is a suffix of $b \Rightarrow (a,b) \in S$.

$b = [1,3] \subseteq [1,4] = c$ and $\min(b) = 1 = \min(c)$. Therefore b is a prefix of $c \Rightarrow (b,c) \in S$. But $\min(a) = 2 \neq 1 = \min(c)$ and $\max(a) = 3 \neq 4 = \max(c) \Rightarrow a$ is neither a prefix nor a suffix of $c \Rightarrow (a,c) \notin S$.

LOGIC & SET THEORY HW 5

OHAD LUTZKY, MAAYAN KESHET

1. Question 1

1.1. B → A.

Proof. Assume there exists a subset $B \subseteq A$ such that $B \sim \mathbb{N}$. Therefore there exists a function $f : \mathbb{N} \to B$ such that f is 1-1 and onto B. Since $B \subseteq A$, then f is privately also a 1-1 function $f : \mathbb{N} \to A$.

□

1.2. A → C.

Proof. Let $f : \mathbb{N} \to A$ be a 1-1 function. Therefore, for any $a \in \text{Range}(f)$, we can uniquely define $f^{-1}(a)$ (since f is 1-1, there exists only one pair (b, a), therefore $f^{-1} = b$ is well-defined). We will therefore define a function $g : A \to A$ that maps any $a \in \text{Range}(f)$ to its "following" element, and any other a to itself. Formally,

$$g(a) = \begin{cases} f(f^{-1}(a) + 1), & a \in \text{Range}(f) \\ a, & a \notin \text{Range}(f) \end{cases} \tag{1}$$

It's easy to see from (**??**) that g is well-defined as a function - for every $a \in A$ we define a unique $g(a)$. Furthermore, g is 1-1: Assume $g(a) = g(b)$. Therefore,

- If $a \notin \text{Range}(f)$, then trivially $g(a) = g(b) = a = b$.
- If $a \in \text{Range}(f)$, then $g(a) = f(\dots)$, therefore also $g(a) \in \text{Range}(f)$. In this case, $g(b) \in \text{Range}(f)$ as well, and thus - by definition of g, $b \in \text{Range}(f)$ (because otherwise, if $b \notin \text{Range}(f)$, then neither is $g(b)$). Therefore we have that $f(f^{-1}(a) + 1) = f(f^{-1}(b) + 1)$, and because f is 1-1, we have that $f^{-1}(a) = f^{-1}(b)$, and then since f is a function, f^{-1} is 1-1, and thus $a = b$.

All that remains is to show that g isn't onto A. We will show that there is no $k \in A$ such that $g(k) = f(0)$. For any $k \in A$,

- If $k \notin \text{Range}(f)$, then $g(k) = k \notin \text{Range}(f)$, and privately $g(k) \neq f(0)$.
- If $k \in \text{Range}(f)$, then $g(k) = f(f^{-1}(k) + 1)$. Seeing as $\text{dom}(f) = \mathbb{N}$, then $f^{-1}(k) \geq 0$, thus $f^{-1}(k) + 1 > 0$, therefore $g(k) \neq f(0)$.

All in all, we've shown a 1-1 function $g : A \to A$ that is not onto A. □

1.3. C → B.

Proof. Assume there exists a function $g : A \to A$ which is 1-1 but not onto A. Therefore exists some $\tilde{a} \in A \setminus \text{Range}(g)$. Define therefore a function $f : \mathbb{N} \to A$ as such:

$$f(i) = \begin{cases} \tilde{a}, & i = 0 \\ g(f(i-1)), & i \geq 1 \end{cases} \tag{2}$$

Now define $B = \text{Range}(f)$. Obviously f is onto B, and since $g : A \to A$, then $B \subseteq A$. All that remains is to show that f is 1-1. We'll prove by induction on i:

Base: ($i = 0$) If $f(0) = f(x)$, then $f(x) = \tilde{a} \notin \text{Range}(g)$, and therefore by (**??**), $x = 0$.

Closure: Assume that if for any x, $f(i) = f(x)$ then $x = i$. Therefore, if $f(i+1) = f(y)$, then $g(f(y-1)) = g(f(i))$, and since g is 1-1, $f(y-1) = f(i)$, and by the inductive assumption, $i = y - 1$, which means that $y = i + 1$.

We've shown a function $f : \mathbb{N} \to B \subseteq A$ such that f is 1-1 and onto B, therefore $B \sim \mathbb{N}$.

□

2. Question 2

2.1. **Part A.** The set is countable. It's obvious that the given set A is of same cardinality as $\mathbb{N} \times \mathbb{N}$, because for each relation R we are given, since it has only one pair, it can be written $\{(a, b)\}$, so we can map using the function $f : A \to \mathbb{N} \times \mathbb{N} : \{(a, b)\} \mapsto (a, b)$. Obviously this function is 1-1 and onto $\mathbb{N} \times \mathbb{N}$, because each pair can be created and different pairs are created by different elements of A. All that remains is to show that $\mathbb{N} \times \mathbb{N}$ is countable. We will write the elements of $\mathbb{N} \times \mathbb{N}$:

$$\begin{matrix} (0,0) & (0,1) & (0,2) & \dots \\ (1,0) & (1,1) & (1,2) & \dots \\ (2,0) & (2,1) & (2,2) & \dots \\ \vdots & \vdots & \vdots & \ddots \end{matrix}$$

We can count members of $\mathbb{N} \times \mathbb{N}$ by following the top-right to bottom-left diagonals. That is, the enumeration is $(0, 0), (0, 1), (1, 0), (0, 2), (1, 1), (2, 0), \dots$. It's clear to see that we arrive at every single pair in $\mathbb{N} \times \mathbb{N}$ in finite time: In level 0, we count $(0, 0)$, in level 1 we count $(0, 1), (1, 0)$, in level i we count $(0, i), (1, i - 1), (2, i - 2), \dots, (i, 0)$ - that is, in level i we count all of the vectors (a, b) such that $a + b = i$. Therefore, we arrive at each (a, b) no later than at level $a + b$, and thus before each element (a, b) we count only a finite number of elements. Thus $\mathbb{N} \times \mathbb{N}$ is countable.

A is also infinite. This is because $f : i \mapsto \{(i, 0)\}$ is clearly a 1-1 function from $\mathbb{N}$ to A.

2.2. **Part B.** The set is countable. We will first count the empty set. Then we will count $\{(0, 0)\}$. Then we will count all of the relations R that, for each pair $(a, b) \in R$, $a + b \leq 1$. At each stage i we will count all of the relations R such that for each pair $(a, b) \in R$, $a + b \leq i$. As we can see from the table in the previous part, that all of the possible pairs in this set are from the triangle between $(i, 0), (0, 0), (0, i)$, and there are $S = \sum_{k=1}^{i} k$ elements in this triangle, and thus 2^S possible relations as such. Since i is finite, so are S and 2^S, and thus at each stage we count only a finite number of elements. For each relation in the set, we are given that it contains a finite number of pairs, therefore, if sorted by sum ($(a, b) \mapsto a+b$), they have a maximum sum $a' + b'$, and thus we will reach them in the finite stage $a' + b'$. Therefore, we reach each relation in the set in a finite number of steps.

The set is also infinite, we can use the same function as in Part A.

2.3. **Part C.** The set is non-countable. This is because each element of it is any possible $R \subseteq \mathbb{N} \times \mathbb{N}$. Therefore this set is precisely $\wp(\mathbb{N} \times \mathbb{N})$. Seeing as $\mathbb{N} \times \mathbb{N}$ is infinite (and countable), then $\wp(\mathbb{N} \times \mathbb{N})$ is, as we've learnt in class, uncountable.

3. Question 3

3.1. **Part A.**

Lemma 1. *If A is countable and F is finite, then $F(A)$ is countable.*

Proof of Lemma **??**. A is countable, therefore $A = \{a_0, a_1, a_2, \dots\}$. F is finite, therefore $F = \{f_1, f_2, f_3, \dots, f_p\}$. We will count the elements of $F(A)$ by function: For each function we will iterate diagonally over possible values of indexes of a. That is, at step j, first we will count all $f_1\left(a_{i_1}, a_{i_2}, \dots, a_{i_{n(f_1)}}\right)$ such that $\sum_{k=1}^{n(f_1)} i_k = j$. We will then do the same for f_2, f_3, and so on until f_p, and then move on to step $j+1$. It's clear that there are a finite number of such vectors for which the sum of the indexes is less than j for any finite j, and since we have a finite number of functions, then each step will count a finite number of elements in $F(A)$, and we've generated all possible values of F resulting from A, thus $F(A)$ has been counted and is, as such, countable.

□

Proof of Part A. We will prove by induction.

Base: For $i = 0$, $D^0 = B$, and as we are given, is countable.

Closure: Assume that D^i is countable. By Lemma **??**, $F(D^i)$ is also countable, and as we've seen in class, a union of two countable sets is countable.

□

3.2. Part B.

Proof. Assume $x \in X_{B,F}$. Therefore, x has a finite creation sequence $\{x_i\}$ such that for each i, either $x_i \in B$ or $x_i = f\left(x_{j_1}, x_{j_2}, \dots, x_{j_{n(f)}}\right)$ such that $f \in F$ and for all k, $j_k < i$. There also exists a finite n such that $x = x_n$. Now, if $x \in B$, then trivially $x \in \bigcup_{i \in \mathbb{N}} D^i$. Otherwise, by the construction of F, for each x_i there exists j such that $x_i \in D^j$. Therefore, there exists such j that all $x_i for$i ¡ n∈ D^j, and therefore $x_n = x \in D^{j+1}$, and thus $x_n \in \bigcup_{i \in \mathbb{N}} D^i$.

Now assume $x \in \bigcup_{i \in \mathbb{N}} D^i$. Therefore there exists such j that $x \in D^j$. By the construction of $F(D^i)$, for every i, D^i is comprised of elements x_i such that either $x_i \in B$ or $x_i = f\left(x_{j_1}, x_{j_2}, \dots, x_{j_{n(f)}}\right)$ such that $f \in F$ and for all k, $j_k < i$. Therefore this holds true for x_n as well, and the relevant x_i are a proper creation sequence for x_i in $X_{B,F}$.

□

3.3. Part C.

We have shown that under the given conditions, D^i is countable for any i. Therefore $\bigcup_{i \in \mathbb{N}} D^i$ is a countable union of countable sets, and as we've shown in class - it is therefore itself countable. And as we've shown, it is equal to $X_{B,F}$, so it, in turn, is also countable.

4. Question 4

4.1. Part A.

The claim is false. Take $A = \mathbb{Z}, C = \mathbb{N}, B = \mathbb{N}, D = \mathbb{Z}$. We've already shown all of these sets to be infinite and countable, thus all of equal cardinality. As we know, $\mathbb{N} \subseteq \mathbb{Z}$, therefore $C \setminus D = \emptyset$, which is finite. However, $A \setminus B = \mathbb{Z} \setminus \mathbb{N} = \mathbb{Z}^-$. We will show that $\mathbb{Z}^- \sim \mathbb{N}$ - take $f : (-z) \mapsto z$. f is trivially 1-1 and onto $\mathbb{N}$, therefore $A \setminus B \sim \mathbb{N} \not\sim \emptyset$, and the claim is false.

4.2. Part B.

The claim is true.

Proof. We know that $A \sim C, B \sim D$. Therefore there exist functions $f : A \to C, g : B \to D$ that are both 1-1 and onto C, D respectively. Consider the function $h : B^A \to D^C$. For every function $x \in B^A$, $h(x) = h_x$ such that $h_x(c) = g(x(f^{-1}(c)))$. It will now suffice to show that h is 1-1, because a function $j : D^C \to B^A$ can be build, and WLOG it will also be 1-1, and by the Cantor-Bernstein theorem we will have cardinality equivelance.

Assume that $h(x) = h(y)$, therefore $h_x = h_y$, which means that for all $c \in C$, $h_x(c) = h_y(c)$. Therefore $g(x(f^{-1}(c))) = g(y(f^{-1}(c)))$. We know that g is 1-1, therefore $x(f^{-1}(c)) = y(f^{-1}(c))$. Since f is 1-1 and onto C, then f^{-1} is onto A, therefore the equality holds for every $a \in A$, so for every a, $x(a) = y(a)$, therefore $x = y$. We have therefore shown one 1-1 function in one direction, and by symmetry we have one in the other, and thus $B^A \sim D^C$.

□

4.3. **Part C.** The claim is true.

Proof. Let us define $f : \left(A^B\right)^C \to A^{(B \times C)}$, such that for all $x \in \left(A^B\right)^C$, $f(x) = f_x : B \times C \to A$, such that $f_x(b, c) = (x(c))(b)$.

f is 1-1: Assume $f(x) = f(y)$, therefore $f_x = f_y$. Thus for all pairs $b, c \in B \times C$, $(x(c))(b) = (y(c))(b)$. Since this holds for every $b \in B$ (because for each such b there is a pair $(b, c) \in B \times C$), then $x(c) = y(c)$. Since *this* holds for every $c \in C$ (same reason), then $x = y$.

Let us define $g : A^{(B \times C)} \to \left(A^B\right)^C$, such that for all $x \in A^{(B \times C)}$, $g(x) = g_x : C \to A^B$, $g_x(c) = g_{x,c} : B \to A$, and $g_{x,c}(a) = (x(b, c))(a)$. We will show that g is 1-1.

Assume $g(x) = g(y)$, therefore $g_x = g_y$. Thus for all $c \in C, g_x(c) = g_y(c)$, so $g_{x,c} = g_{y,c}$. Therefore for all $a \in A$, $g_{x,c}(a) = g_{y,c}(a)$. So we have that for all $a, b, c \in A, B, C, (x(b, c))(a) = (y(b, c))(a)$, and this is only possible if for all $(b, c) \in B \times C, x(b, c) = y(b, c)$, so $x = y$.

We've shown a 1-1 function in each direction, so by the Cantor-Bernstein theorem, the sets are of equal cardinality.

□

4.4. **Part D.** The claim is false. Assume $B = C = 0, A = 0, 1$. Then A^B has exactly two functions - constant 0 and constant 1, that is, $A^B = \{f_0, f_1\}$. Since $B = C$, also $A^C = \{f_0, f_1\}$, and thus $A^B \times A^C = \{(f_0, f_0), (f_0, f_1), (f_1, f_1), (f_1, f_0)\}$, and $\left|A^B \times A^C\right| = 4$. However, $B = C$, therefore $B \cup C = B$, and thus $A^{B \cup C} = A^B$, so as we've shown, $\left|A^{B \cup C}\right| = \left|A^B\right| = 2 \neq 4$.

5. Question 5

5.1. ***A*.** A is uncountable.

Proof. Assume by contrast that A is countable. Therefore there exists $f : \mathbb{N} \to A$ which is 1-1 and onto A. Also, let $B_\heartsuit$ the set of infinte binary vectors with an infinite number of 1s and an infinite number of 0s. We will show a 1-1 function from $B_\heartsuit$ onto A:

$k : A \to B_\heartsuit$ will be defined as $k(X) = b$ such that $b_i = 1 \iff i \in X$. Because X is infinite, and for each $i \in X$, $b_i = 1$, then b has an infinite number of 1s. Because $\mathbb{N} \setminus X$ is infinite, and for each $i \in \mathbb{N} \setminus X, i \notin X$ then $b_i = 0$, then b has an infinite number of 0s. Therefore $b \in B_\heartsuit$. Clearly this function is 1-1, because if WLOG $a \in X_1, a \notin X_2$, then $f(X_1)_a = 1 \neq 0 = f(X_2)_a$. It is also onto B because any vector $B \in B_\heartsuit$ can be represented by an appropriate set X for which every i that $b_i = 1$ maintains $i \in X$. Again, by the same argument, since b has infinite 1s and 0s, both X and $\mathbb{N} \setminus X$ will be infinite.

Now, we've assumed that f is 1-1 and onto A, and proven that k is 1-1 and onto $B_\heartsuit$. Therefore $h = f \circ k$ is 1-1 and onto $B_\heartsuit$. Examine the values of h: (We don't know what they are, because f is unknown. We do know they're binary vectors though)

$$\begin{array}{rcl} h(0) & = & \mathbf{b_{00}}b_{01}b_{02}b_{03}b_{04}b_{05}b_{06}b_{07}\dots \\ h(1) & = & b_{10}b_{11}b_{12}\mathbf{b_{13}}b_{14}b_{15}b_{16}b_{17}\dots \\ h(2) & = & b_{20}b_{21}b_{22}b_{23}b_{24}b_{25}\mathbf{b_{26}}b_{27}\dots \\ & \vdots & \end{array}$$

Consider the following vector h^*:

$$h^* = \overline{b_{00}}01\overline{b_{13}}01\overline{b_{26}}01\dots$$

As we've shown, h is onto $B_\heartsuit$. $h^* \in B_\heartsuit$, seeing as it clearly has an infinite number of 0s and 1s. Therefore there exists i such that $h^* = h(i)$. However, $h(i)_{3i} = b_{i,3i}$, whereas $h^*_{3i} = \overline{b_{i,3i}}$, therefore for any $i \in \mathbb{N}$, $h^* \neq h(i)$. This is in contradiction to h being onto $B_\heartsuit$, which is only possible if our original assumption that f is onto A was false. Therefore A cannot be countable.

□

5.2. B. B is countable.

Proof. We will use the same function k we've defined before, only this time it will have the domain B, and the range $B_\spadesuit$, which will be the binary vectors with a finite number of 0s. Because of the same arguments as before, k will be 1-1 and onto $B_\spadesuit$ - 0s are for $i \in \mathbb{N} \setminus X$, and there are a finite number of those.

Therefore, $B \sim B_\spadesuit$. All that remains is to show that $B_\spadesuit$ is countable. We can do this by counting the negatives in ordinary binary order, "starting from the end", that is - $11111\dots, 01111\dots, 10111\dots, 00111\dots, 11011\dots, 01011\dots, \dots$. Each vector with a finite number of 0s has a maximal index

$$i_M = \text{argmax}_{i\in\mathbb{N}}(b_i = 0)$$

Therefore the vector $\underbrace{1111\dots1}_{\times i_M+1}0111\dots$ will be counted after it, and will be counted at step 2^{i_M+1}, then all vectors with a finite number of 0s are reached in a finite number of steps.

□

LOGIC AND SET THEORY HW 6

OHAD LUTZKY, MAAYAN KESHET

1. Question 1

Claim 1. *Let $X_{B,F} \subseteq Z$ be an inductively defined group, and $x \in Z$. Then $x \in X_{B,F}$ iff x has a creation sequence in $X_{B,F}$.*

Because **WFF** was defined inductively as a subset of $(Symb \cup Var)^*$, then the claim immediately answers question 1.

Proof of Claim **??**. First direction: By structure induction. Let

$$Y = \{z \in Z | z \text{ has a creation sequence in } X_{B,F}\}$$

Then we will show that $X_{B,F} \subseteq Y$.

Basis: All $y \in B$ have a trivial finite creation sequence:

$$y \quad (Base)$$

Closure: We will show that Y is closed under F. Assume $f_i \in F$ is an m-valued function, $y_1, \ldots, y_m \in Y$, then $y_1, \ldots, y_m$ each have some creation sequence $s(y_j)$. As we've shown in a previous homework exercise, concatenation of creation sequences yields a valid creation sequence. Therefore, we will take the concatenation $s(y_1)|s(y_2)| \ldots |s(y_m)|f_i(y_1, \ldots, y_m)$. This is a valid creation sequence — from the first entry in $s(y_1)$ to the last entry of $s(y_m)$ we have already shown validity, and the new entry $f(\ldots)$ is valid because it is a function of $y_1, \ldots, y_m$, all of which are previous entries in the creation sequence.

This creation sequence is finite because $s(y_j)$ are all, by the inductive assumption, finite, and we've only added 1 entry.

Second direction: By induction on the length of the creation series.

For the case where the length of the creation series is 1, we have already shown in a previous exercise that the creation series must be a single element of B, and is thus trivially a member of $X_{B,F}$.

Now, assume the claim is true for all creation series of length $\leq k$, and we will show for length $k+1$. Let $s_1, s_2, \ldots, s_k, s_{k+1}$ be a creation series. Then each prefix $s_1, \ldots, s_j$ such that $j \leq k$ is a creation series (we've shown prefixes of creation sequences to be themselves valid creation sequences) of length $j \leq k$, therefore by the inductive assumption, $s_1, \ldots, s_k \in X_{B,F}$. Now, seeing as $s_1, \ldots, s_k, s_{k+1}$ is also a valid creation sequence, then there are two options: If $s_{k+1} \in B$, then trivially $s_{k+1} \in X_{B,F}$. Therefore we only need to show for the case that $s_{k+1} = f_i(s_{j_1}, s_{j_2}, \ldots, s_{j_m})$ where $f_i \in F$ is an m-valued function. But this is also trivial, seeing as by definition, $X_{B,F}$ is closed under F. □

2. Question 2

2.1. Part A.

Proof. Let *validpar* be the property described — i.e., $validpar(\varphi)$ means that between any pair of parentheses of the form $)w($ in φ, w contains at least one connector. Formally, if we enumerate all parentheses in φ like so — $\varphi = (_0(_1)_2)_3(_4)_5$, and let $\#_{()}(\varphi)$ be their count (6 in this case), then for all $i < \#_{()}$ such that $)_i$ is in φ (that is, the ith bracket is a closing bracket), then between it and $(_{i+1}$ there is a connector.

Let $Y = \{\varphi \in (Symb \cup Var)^* | validpar(\varphi)\}$. We will show by structure induction on **WFF** that **WFF** $\subseteq Y$.

Basis: For each $i \in \mathbb{N}$, p_i has no parentheses, then the claim is trivially held for those. Identically, it holds for **T** and **F**.

Closure: Assume $\varphi_1, \varphi_2 \in Y$, and we will show that $f_\neg(\varphi_1), f_\circ(\varphi_1, \varphi_2) \in Y$. The claim is trivial for $f_\neg(\varphi_1) = \neg\varphi_1$ — we haven't added any new parentheses, and the claim already holds (by assumption) for φ_1.

As for $f_\circ(\varphi_1, \varphi_2) = (\varphi_1 \circ \varphi_2)$, we must check for every closing bracket, that between it and the nearest following open bracket there is a connector.

Let $)_i$ be a closing bracket in φ_1 (if any exist). By the assumption, either there is a connector between $)_i$ and $(_{i+1}$, or there is no $(_{i+1}$ in φ_1. In this case, the first following opening bracket, if any, will be in φ_2 — and this will follow the connector $\circ$.

For every closing bracket in φ_2, again, since φ_2 maintains the inductive assumption, then each closing bracket in φ_2 is either followed by no opening bracket at all (not in φ_2, and we haven't added any), or is followed by a connector first.

□

2.2. **Part B.**

Proof. Let *onemorevar* be the property described — i.e., $onemorevar(\varphi)$ means that $\#_{var}(\varphi) = \#_{con2}(\varphi)$. Let $Y = \{\varphi \in (Symb \cup Var)^* | onemorevar(\varphi)\}$, and we will show that **WFF** $\subseteq Y$ by structure induction.

Basis: For all atomic formulae $\varphi \in$ **WFF**, $\#_{var}(\varphi) = 1$ whereas $\#_{con2}(\varphi) = 0$, so the claim holds.

Closure: We need to show that Y is closed under the following functions:
- Assuming $\varphi \in Y$, we can see that $\neg\varphi$ maintains $\#_{var}(\varphi) = \#_{var}(\neg\varphi)$, $\#_{con2}(\varphi) = \#_{con2}(\neg\varphi)$, as we've only added one connector which is unary, therefore $\neg\varphi \in Y$ as well.
- Assuming $\varphi_1, \varphi_2 \in Y$, we have by definition of Y that $\#_{var}(\varphi_1) = \#_{con2}(\varphi_1) + 1$, and $\#_{var}(\varphi_2) = \#_{con2}(\varphi_2) + 1$. Examine $\varphi_1 \circ \varphi_2$. It has all of the variables of φ_1 and φ_2, with no added variables, therefore $\#_{var}(\varphi_1 \circ \varphi_2) = \#_{var}(\varphi_1) + \#_{var}(\varphi_2)$. But by the assumption, this is equal to $\#_{con2}(\varphi_1) + 1 + \#_{con2}(\varphi_2) + 1$. The number of binary connectors in $\varphi_1 \circ \varphi_2$ is, plainly, $\#_{var}(\varphi_1) + \#_{var}(\varphi_2) + 1$ (the $\circ$ causing the $+1$), so we have that $\#_{var}(\varphi_1 \circ \varphi_2) = \#_{con2}(\varphi_1 \circ \varphi_2) + 1$.

□

3. Question 3

3.1. **Part A.** The claim is false. Take the example $\varphi = \to p_0 \to\to p_0 p_0 p_0$ - we must show that $\varphi \in POL$, and that the longest chain of binary connectors in φ is not a prefix of φ. The latter is trivial — the longest chain of connectors in φ is $\to\to$, which is clearly not a prefix of φ. All that remains is to show a creation sequence for φ over POL, and by claim **??** we will have $vp \in POL$, thus φ will be a less counter example to the claim.

The following creation sequence will be appropriate:

1.	p_0	(base)
2.	p_0	(base)
3.	$\to p_0p_0$	$(\to 1, 2)$
4.	$\to\to p_0p_0$	$(\to 3, 1)$
5.	$\to p_0 \to\to p_0p_0p_0$	$(\to 1, 4)$

3.2. **Part B.**

Claim 2. *If $\varphi \in POL$, then $\#_{var}(\varphi) = \#_{con2}(\varphi) + 1$.*

Claim 3. *If $\psi \in POL$, and φ is a proper*[1] *prefix of ψ, then $\#_{var}(\varphi) \neq \#_{con2}(\varphi)+1$.*

Proof. Proof of Claim **??** Let $Y = \{\psi \in (Symb \cup Var)^* | \#_{var}(\psi) = \#_{con2}(\psi) + 1\}$. We will show by strctural induction that $POL \subseteq Y$, and therefore for any $\psi \in POL$, $\#_{var}(\psi) = \#_{con2}(\psi) + 1$.

Basis: For any atom $p_i \in Var$, $\#_{var}(p_i) = 1$, and $\#_{con2}(p_i) = 0$, therefore the property is maintained. The same holds for $\mathbf{T}, \mathbf{F}$.

Closure: We have to prove for both the unary and binary operations:

- Assume $\varphi \in Y$, and examine $\neg\varphi$. Clearly we have added nothing but a unary connector, and removed nothing, thus
$$\#_{var}(\neg\varphi) = \#_{var}(\varphi), \#_{con2}(\neg\varphi) = \#_{var}(\varphi)$$
By the inductive assumption $\#_{var}(\neg\varphi) = \#_{con2}(\neg\varphi) + 1$, therefore $\neg\varphi \in Y$.
- Assume $\varphi, \psi \in Y$, and examine $\alpha = \circ\varphi\psi$. We have clearly retained all previous variables and binary connectors, and added one. Thus, $\#_{con2}(\alpha) = 1 + \#_{con2}(\varphi) + \#_{con2}(\psi)$ and $\#_{var}(\alpha) = \#_{var}(\varphi) + \#_{var}(\psi)$. But by the inductive assumption,
$$\#_{var}(\varphi) + \#_{var}(\psi) = \#_{con2}(\varphi) + \#_{con2}(\psi) + 2 = \#_{con2}(\alpha) + 1$$
Therefore $\alpha \in Y$.

We have shown that $POL \subseteq Y$. □

Proof. Proof of Claim **??** Let $lackingprefix$ be the described property — that is, $lackingprefix(\psi)$ means that if φ is a proper prefix of ψ, then $\#_{var}(\varphi) < \#_{con2}(\varphi) + 1$. Let $Y = \{\psi \in POL | lackingprefix(\psi)\}$, then we will show that $Y \subseteq POL$ by structural induction. Note that we assume $Y \subseteq POL$, therefore we will have $Y = POL$.

Basis: All atoms p_i, as well as $\mathbf{T}, \mathbf{F}$, have no proper prefixes, therefore the property holds trivially.

Closure: We have to prove for both the unary and binary operations:

- Assume $\psi \in Y$, and examine $\neg\psi$. Then there are two options for a proper prefix:
  - If the proper prefix is simply $\neg$, then obviously $\#_{var}(\neg) = 0 < 1 = \#_{con2}(\neg) + 1$.
  - Any other proper prefix φ' of $\neg\psi$ can clearly be written as $\neg\varphi$, φ being a proper prefix of ψ. By the assumption, $\psi \in Y$ and therefore $\#_{var}(\varphi) < \#_{con2}(\varphi) + 1$. But, once again, $\#_{var}(\varphi) = \#_{var}(\neg\varphi), \#_{con2}(\varphi) = \#_{con2}(\neg\varphi)$, and all in all - $lackingprefix(\neg\psi)$. Therefore $\neg\psi \in Y$.
- Assume $\psi_1, \psi_2 \in Y$, and examine $\circ\psi_1\psi_2$. Let φ be a proper prefix of $\circ\psi_1\psi_2$, then there are the following options:
  - If $\varphi = \circ$, then $\#_{var}(\circ) = 0 < \#_{con2}(\circ) + 1 = 2$. Then $\varphi \in Y$.

[1]We will say that φ is a *proper prefix* of ψ if $\varphi \neq \sigma$, $\varphi \neq \psi$, and φ is a prefix of ψ.

- If $\varphi = \circ\varphi_1$, φ_1 being a proper prefix of ψ_1, then obviously $\#_{var}(\varphi) = \#_{var}(\varphi_1)$, and $\#_{con2}(\varphi) = \#_{con2}(\varphi_1) + 1$. By the inductive assumption, $lackingprefix(\psi_1)$, therefore $\#_{var}(\varphi_1) < \#_{con2}(\varphi_1) + 1$. All in all, we have that

$$\#_{var}(\varphi) = \#_{var}(\varphi_1) < \#_{con2}(\varphi_1) + 1 = \#_{con2}(\varphi) < \#_{con2}(\varphi) + 1$$

- If $\varphi = \circ\psi_1$, then $\#_{var}(\varphi) = \#_{var}(\psi_1)$, $\#_{con2}(\varphi) = \#_{con2}(\psi_1) + 1$. But $\psi_1 \in Y$, therefore $\psi_1 \in POL$, and by Claim **??**, $\#_{var}(\psi_1) = \#_{con2}(\psi_1) + 1$. All in all, we have that

$$\#_{var}(\varphi) = \#_{var}(\psi_1) = \#_{con2}(\psi_1) + 1 = \#_{con2}(\varphi) < \#_{con2}(\varphi) + 1$$

- If $\varphi = \circ\psi_1\varphi_2$, φ_2 being a proper prefix of ψ_2, then $\#_{var}(\varphi) = \#_{var}(\psi_1) + \#_{var}(\varphi_2)$, $\#_{con2}(\varphi) = 1 + \#_{con2}(\psi_1) + \#_{con2}(\varphi_2)$. Again, $\psi_1 \in POL$, therefore $\#_{var}(\psi_1) = \#_{con2}(\psi_1) + 1$, and $lackingprefix(\psi_2)$, thus $\#_{var}(\varphi_2) < \#_{con2}(\varphi_2) + 1$. All in all,

$$\begin{aligned}
\#_{var}(\varphi) &= \\
&= \#_{var}(\psi_1) + \#_{var}(\varphi_2) \\
&= \#_{con2}(\psi_1) + 1\#_{var}(\varphi_2) \\
&< \#_{con2}(\psi_1) + \#_{con2}(\varphi_2) + 1 + 1 \\
&= \#_{con2}(\varphi) + 1
\end{aligned}$$

We have shown that $Y = POL$, therefore for every prefix φ of a prefix formula $\psi \in POL$, $\#_{var}(\varphi) < \#_{con2}(\psi) + 1$. □

Proof of **??**. Assume $\varphi, \psi \in POL$, and that φ is a prefix of ψ. We need to show that $\varphi = \psi$. Assume by contrast that $\varphi \neq \psi$, then by Claim **??**, $\#_{var}(\varphi) \neq \#_{con2}(\varphi)+1$, and then by reversal of Claim **??**, $\varphi \notin POL$. □

3.3. **Part C.**

Proof. Let X be either POL or **WFF**. In either case, X is infinite: $Var \subseteq X$, and Var is infinite. Furthermore, X is countable: X is, in both cases, an inductive set with a countable basis (Var is defined as an enumeration of the atomic formulae p_i, and the addition of $\mathbf{T}, \mathbf{F}$, by the infinite hotel theorem, keeps it countable), and a finite closure ($|F| = 4$ in both cases), and thus by a theorem we've shown in HW 5, X is countable.

We've shown both POL and **WFF**to be infinite and countable. Thus we have $POL \sim \mathbb{N}, \mathbf{WFF} \sim \mathbb{N}$, and therefore $POL \sim \mathbf{WFF}$. □

4. Question 4

4.1. **Part A.**

Proof. We need to show that **WFF** is closed under the *subst* function. We will show this by structure induction:

Basis: If $\varphi = p_i$, then for any substitution s, $subst(\varphi, s) = s(p_i)$. By definition, $s(p_i) \in \mathbf{WFF}$.

If $\varphi \in \mathbf{T}, \mathbf{F}$, then for any substitution s, $subst(\varphi, s) = \varphi$, and by the assumption $\varphi \in \mathbf{WFF}$.

Closure: We need to show that **WFF** is closed under *subst*, for both binary and unary functions on formulae in **WFF**:

- Assume $\varphi \in \mathbf{WFF}$, and that for any substitution s, $subst(\varphi, s) \in \mathbf{WFF}$. Then $subst(\neg\varphi, s) = \neg subst(\varphi, s)$, and since $subst(\varphi, s) \in \mathbf{WFF}$, by definition of **WFF**, $\neg subst(\varphi, s) \in \mathbf{WFF}$.

- Assume $\varphi_1, \varphi_2 \in \mathbf{WFF}$, and that for any $s : Var \to \mathbf{WFF}$, both $subst(\varphi_1, s) \in \mathbf{WFF}$ and $subst(\varphi_2, s) \in \mathbf{WFF}$. Then

  $$subst((\varphi_1 \circ \varphi_2), s) = (subst(\varphi_1, s) \circ subst(\varphi_2, s))$$

  By definition of **WFF**, since by the assumption both $subst(\varphi_1, s)$ and $subst(\varphi_2, s) \in \mathbf{WFF}$, then so is $(subst(\varphi_1, s) \circ subst(\varphi_2, s))$.

□

4.2. **Part B.** The claim is false. Take $s = s_{\mathbf{T}}$, that is, $s(p_i) = \mathbf{T}$ for any natural i, and take $t = I$, that is, $s(p_i) = p_i$ for any natural i. Then take $\varphi = \mathbf{T}$. By definition of $subst$, $subst(\varphi, s) = subst(\varphi, t) = \mathbf{T}$, yet clearly $s \neq t$.

4.3. **Parts C,D.**

Definition. *Let* $\mathcal{P}_{i=0}^{n} = \{p_0, p_1, p_2, \dots, p_n\}$. *Let* $\mathcal{P}_{i=0}^{\infty} = \{p_0, p_1, p_2, p_3, \dots\} = Var$.

Claim 4. *For any natural n or $n = \infty$,* $subst_2(\mathcal{P}_{i=0}^{n}, s_{\boldsymbol{T}}) = \{\boldsymbol{T}\}$.

Proof of Claim **??**. Assume $\varphi \in subst_2(\mathcal{P}_{i=0}^{n}, s)$. Then there exists $\psi \in \mathcal{P}_{i=0}^{n}$ such that $\varphi = subst(\psi, s_{\mathbf{T}})$. But by definition of $\mathcal{P}_{i=0}^{n}$, the only possible values for ψ are p_i, and $subst(p_i, s_{\mathbf{T}}) = s_{\mathbf{T}}(p_i) = \mathbf{T}$ for any p_i of these. Then $\varphi = \mathbf{T}$, so $subst_2(\mathcal{P}_{i=0}^{n}, s) \subseteq \{\mathbf{T}\}$. As we've shown, $\mathbf{T} \in subst_2(\mathcal{P}_{i=0}^{n}, s)$, therefore $\{\mathbf{T}\} \subseteq subst_2(\mathcal{P}_{i=0}^{n}, s)$. □

Both claims C and D are false.

Counterexample for Part C: Take $s = s_{\mathbf{T}}, \Sigma = \mathcal{P}_{i=0}^{42}$. Clearly, Σ is finite, and furthermore $|\Sigma| = 42$. However, by Claim **??**, $|subst_2(\Sigma, s)| = 1$, thus $\Sigma \not\sim subst_2(\Sigma, s)$.

Counterexample for Part D: Take $s = s_{\mathbf{T}}, \Sigma = \mathcal{P}_{i=0}^{\infty}$. As we've shown in class, $\Sigma = Var$ is infinite. However, by Claim **??**, $|subst_2(\Sigma, s)| = 1$, thus $\Sigma \not\sim subst_2(\Sigma, s)$.

LOGIC & SET THEORY HW 7

OHAD LUTZKY

2. Question 2

2.1. Part A.

Proof. **Basis:** For $k = 0$, $\Sigma = \emptyset$, then $\bigvee \Sigma = \mathbf{F}$. Take any assignment z, then it trivially does not satisfy $\bigvee \Sigma$. Also, trivially there does not exist $\varphi \in \Sigma$ which z satisfies.

Closure: Assume the claim holds for $|\Sigma| = k$, we'll show it for $|\Sigma| = k + 1$.

First direction: Assume there exists $\varphi \in \Sigma$ such that $z \vDash \varphi$. Seeing as $\Sigma = \{\varphi_0, \ldots, \varphi_{k-1}, \varphi_k\}$, either $\varphi = \varphi_k$ or $\varphi = \varphi_i$ where $i < k$. Assume the former, then by $TT_\vee$, z must satisfy $\bigvee \Sigma = (\bigvee\{\varphi_0, \ldots, \varphi_{k-1}\} \vee \varphi_k)$. If we assume the latter, then by the inductive assumption, since there exists $\varphi_i \in \{\varphi_0, \ldots, \varphi_{k-1}$ which z satisfies, then z satisfies $\bigvee\{\varphi_0, \ldots, \varphi_{k-1}\}$, and thus by $TT_\vee$ it satisfies $\bigvee \Sigma$.

Second direction: Assume that z satisfies $\bigvee \Sigma$. Then by $TT_\vee$, it either satisfies φ_k or it satisfies $\bigvee\{\varphi_0, \ldots, \varphi_{k-1}\}$ (or both). If we assume the former, then we're done - we've found a formula in Σ which z satisfies. Assume then, that z does not satisfy φ_k. Then by $TT_\vee$, as we've said, it must satisfy $\bigvee\{\varphi_0, \ldots, \varphi_{k-1}\}$. But by the inductive assumption, this means that there exists φ_i with $i < k$ such that z satisfies φ_i. Obviously, $\varphi_i \in \Sigma$, and we're done.

□

2.2. Part B.

Proof. **Basis:** For $k = 0$, $\Sigma = \emptyset$, then $\bigwedge \Sigma = \mathbf{T}$. Take any assignment z, then it trivially satisfies $\bigwedge \Sigma$. Also, trivially it satisfies every formula in Σ, so $z \vDash \Sigma$.

Closure: Assume the claim holds for $|\Sigma| = k$, we'll show it for $|\Sigma| = k + 1$.

First direction: Assume that $z \vDash \Sigma$. Then for every $\varphi \in \Sigma$, z satisfies φ. Privately, z also satisfies $\{\varphi_0, \ldots, \varphi_{k-1}\}$, and thus by the inductive assumption it satisfies $\bigwedge\{\varphi_0, \ldots, \varphi_{k-1}\}$. Also, it privately satisfies φ_k. Thus, by $TT_\wedge$, it satisfies $\bigwedge \Sigma$.

Second direction: Assume that z satisfies $\bigwedge \Sigma$. Then by $TT_\wedge$, it both satisfies φ_k and $\bigwedge\{\varphi_0, \ldots, \varphi_{k-1}\}$. By the inductive assumption, this means that it also satisfies $\{\varphi_0, \ldots, \varphi_{k-1}\}$, and altogether we've shown that it satisfies every formula in Σ, that is, $z \vDash \Sigma$.

□

2.3. Part C.

Proof. Assume z satisfies $\bigwedge_{i=0}^{k-1}(\neg\varphi_i)$. Then by Part B, it satisfies $\neg\varphi_i$ for all $i < k$. By $TT_\neg$, that means that it does not satisfy φ_i for all $i < k$, and then by Part A, that means that it does not satisfy $\bigvee_{i=0}^{k-1} \varphi_i$. But, again by $TT_\neg$, we have that z satisfies $\neg \bigvee_{i=0}^{k-1} \varphi_i$.

Reversal of the proverbial arrows will give us the other direction, and thus we have shown logical equivelance of the two formulae. □

2.4. **Part D.**

Proof. Assume z satisfies $\neg \bigwedge_{i=0}^{k-1} (\neg\varphi_i)$. Then by $TT_\neg$, it does not satisfy $\bigwedge_{i=0}^{k-1} (\neg\varphi_i)$. By Part B, this means that there exists φ_i with $i < k$ such that z does not satisfy φ_i. Therefore, by $TT_\neg$, there exists φ_i such that z does satisfy $\neg\varphi_i$, and then by Part A, this means that z satisfies $\bigvee_{i=0}^{k-1} (\neg\varphi_i)$.

Reversal of the proverbial arrows will give us the other direction, and thus we have shown logical equivelance of the two formulae. □

3. Question 3

3.1. **Part A.** The claim is false. $\emptyset$ is trivially, antisymmetric with respect to any assignment, but is also emptily satisfiable by any assignment.

3.2. **Part B.** The claim is false. Take $\Sigma_1 = \emptyset, \Sigma_2 = \{\varphi_0\}$. As in part A, $L(\Sigma_1) = ASS$, but for any assignment z, there does not exist $\alpha \in \Sigma_2$ such that $\bar{z}(\alpha) \neq \bar{z}(\varphi_0)$, since only $\varphi_0 \in \Sigma_2$, thus $L(\Sigma_2) = \emptyset$. In summary, $L(\Sigma_1) = ASS, L(\Sigma_2) = \emptyset, \Sigma_1 \cup \Sigma_2 = \Sigma_2, L(\Sigma_2) = L(\Sigma_2) = \emptyset \neq L(\Sigma_1) \cup L(\Sigma_2) = ASS$.

4. Question 4

4.1. **Part A.**

Lemma 1 (The Chocolate Chip Cookie lemma)**.** *If $A, B \in \wp(\mathbf{WFF}), \alpha \in \mathbf{WFF}$, and $A \cap B \vDash \alpha$, then $A \vDash \alpha$ and $B \vDash \alpha$.*

Proof of Lemma **??**. It suffices to show that $A \vDash \alpha$, and then symmetrically, $B \vDash \alpha$. We must therefore show that for each $z \in ASS$, $z \vDash A \Rightarrow z \vDash \alpha$[1]. But $z \vDash A$ means that for any $\varphi \in A$, $z \vDash \varphi$. Privately, this holds for $\varphi \in A \cap B \subseteq A$, therefore $z \vDash A \cap B$. But by the assumption, this means that $z \vDash \alpha$. Thus $A \vDash \alpha$. □

Proof of 4A. Assume T, T' are theories. If $T \cap T' \vDash \alpha$, then by the Chocolate Chip Cookie Lemma (**??**), both $T \vDash \alpha$ and $T' \vDash \alpha$. But T, T' are theories, thus $\alpha \in T, \alpha \in T'$, or in other words (symbols), $\alpha \in T \cap T'$. We have shown that if $T \cap T' \vDash \alpha$, then $\alpha \in T \cap T'$, so $T \cap T'$ is a theory. □

4.2. **Part B.**

Proof. Assume by contrast that neither $T \subseteq T'$ nor $T' \subseteq T$. Therefore exist $\alpha \in T \setminus T', \beta \in T' \setminus T$, and thus $\alpha, \beta \in T \cup T'$. Then any assignment which satisfies $T \cup T'$ would have to satisfy α, β, and so by $TT_\wedge$, it satisfies $\alpha \wedge \beta$, or in other words — $T \cup T' \vDash \alpha \wedge \beta$. But $T \cup T'$ is a theory, so $\alpha \wedge \beta \in T \cup T'$, meaning $\alpha \wedge \beta \in T$ or $\alpha \wedge \beta \in T'$. Assume the former, then any assignment which satisfies T must satisfy $\alpha \wedge \beta$, and by $TT_\wedge$, to do this it must satisfy β, meaning $T \vDash \beta$. Thus $\beta \in T$, in contrast to the assumption. If we assume the latter, that is, $\alpha \wedge \beta \in T'$, then we identically reach the conclusion that $\alpha \in T'$, again in contrast to the assumption. Thus either $T \subseteq T', or T' \subseteq T$. □

[1]We denote z satisfies $\varphi \in \mathbf{WFF}$ or z satisfies $\Sigma \in \wp(\mathbf{WFF})$ by $z \vDash \varphi, z \vDash \Sigma$ respectively.

5. Question 5

5.1. **Part A.** The claim is true.

Proof. Let z be the said assignment. φ_z depends on k, so we will call it $\varphi_{z,k}$ and define it inductively.

Basis: $\varphi_{z,0} = \mathbf{T}$

Closure: $\varphi_{z,i+1} = \begin{cases} (p_i \wedge \varphi_{z,i}), & z(p_i) = 1 \\ (\neg p_i \wedge \varphi_{z,i}), & z(p_i) = 0 \end{cases}$

We will now prove that such φ_z maintains the claim.

First direction: Clearly $\varphi_{z,k}$ only holds the variables $p_0, \ldots, p_{k-1}$, thus when evaluating the meaning - seeing as z, z' are equal with respect to their assignments on $p_0, \ldots, p_{k-1}$, we will reach the same meaning. All that is left to show is that z satisfies $\phi_{z,k}$, because then so does z'.

Basis: For $k = 0$, any z trivially satisfies $\varphi_{z,0}$.

Closure: Assume that z satisfies $\varphi_{z,k}$, and we'll show

again, inductively.

Basis: For $k = 0$, trivially, any two assignments z, z' are equal with respect to their assignments on $p_0, \ldots, p_{k-1}$, thus any z' must satisfy the formula. But the formula is $\mathbf{T}$, so it does.

Closure: Assume that z satisfies $\varphi_{z,k}$, and we'll show that it satisfies $\varphi_{z,k+1}$. If $z(p_k) = 1$, then

$$\begin{aligned} M(\varphi_{z,k+1}, z) &= M((p_k \wedge \varphi_{z,k})), z) \\ &= TT_\wedge(M(p_k, z), M(\varphi_{z,k}, z)) \end{aligned}$$

But by the inductive assumption, $M(\varphi_{z,k}, z) = 1$, so

$$= 1$$

Thus z satisfies $\varphi_{z,k+1}$. If $z(p_k) = 0$, then

$$\begin{aligned} M(\varphi_{z,k+1}, z) &= M((\neg p_k \wedge \varphi_{z,k})), z) \\ &= TT_\wedge(M(\neg p_k, z), M(\varphi_{z,k}, z)) \\ &= TT_\wedge(TT_\neg(p_k, z), M(\varphi_{z,k}, z)) \\ &= TT_\wedge(1, M(\varphi_{z,k}, z)) \end{aligned}$$

But by the inductive assumption, $M(\varphi_{z,k}, z) = 1$, so

$$= 1$$

Second direction: We have to show that if z' satisfies $\varphi_{z,k}$, then it identifies with z on variables $p_0, \ldots, p_{k-1}$.

Basis: For $k = 0$, trivially, any assignment satisfies $\varphi z, 0 = \mathbf{T}$. But also trivially, any two assignments z, z' are equal with respect to their assignments on $p_0, \ldots, p_{k-1}$.

Closure: Assume that z' satisfies $\varphi_{z,k+1}$, and that it identifies with z on p_i for $i < k$, and we'll show that it identifies with z on p_k. Assume that $z(p_k) = 1$, we'll show that $z'(p_k) = 1$.

$$\begin{aligned} 1 = M(\varphi_{z,k+1}, z') &= M((p_k \wedge \varphi_{z,k})), z') \\ &= TT_\wedge(M(p_k, z'), M(\varphi_{z,k}, z')) \end{aligned}$$

Therefore, by $TT_\wedge$, $M(p_k, z') = 1$. Now assume that $z(p_k) = 0$, and we'll show that $z'(p_k) = 0$.

$$\begin{aligned}1 = M(\varphi_{z,k+1}, z') &= M((\neg p_k \wedge \varphi_{z,k})), z') \\ &= TT_\wedge(M(\neg p_k, z'), M(\varphi_{z,k}, z')) \\ &= TT_\wedge(TT_\neg(M(p_k, z')), M(\varphi_{z,k}, z'))\end{aligned}$$

Therefore, by $TT_\wedge$, we have that $TT_\neg(M(p_k, z')) = 1$, so by $TT_\neg$ we have that $z'(p_k) = 0$.

We have shown, for every $z \in ASS, k \in \mathbb{N}$, a formula $\varphi_{z,k} \in \mathbf{WFF}(k)$ for which z' satisfies $\varphi_{z,k}$ iff z, z' are identical with respect to their assignments on $p_0, \ldots, p_{k-1}$. □

5.2. **Part B.**

Proof. As we've shown previously, $\mathbf{WFF} \sim \mathbb{N}$. By definition, $\mathbf{WFF}(k) \subseteq \mathbf{WFF}$, and as we've shown in class, this means $\mathbf{WFF}(k) \preceq \mathbf{WFF}$. We will show that $\mathbb{N} \preceq \mathbf{WFF}(k)$, and thus by the Cantor-Bernstein theorem, $\mathbf{WFF}(k) \sim \mathbf{WFF}$. We need to show a 1-1 function from $\mathbb{N}$ to $\mathbf{WFF}(k)$. This is simple enough: Take

$$f(i) = \underbrace{\neg\neg\neg\ldots\neg}_{\times i}\mathbf{T}$$

This function is clearly 1-1. Also, the expression given is within $\mathbf{WFF}(k)$ since it doesn't use any variables. □

5.3. **Part C.**

5.3.1. *Part i.*

Definition 1. *Let* $ASS(k) = \{z \in ASS | z(p_i) = 0 \textit{ for all } i \geq k\}$

Definition 2. *For any* $Z \in \wp(ASS(k))$*, define* $\Phi_Z = \{\varphi_z \in \mathbf{WFF}(k) | z \in Z\}$.

Definition 3. $\Sigma_M = \{\bigvee \Phi_Z \in \mathbf{WFF}(k) | Z \in \wp(ASS(k))\}$

5.3.2. *Part ii.*

Proof. Let $Z_1, Z_2 \in \wp(ASS(k)), Z_1 \neq Z_2$. Then we will show that $\bigvee \Phi_{Z_1} \not\sim \bigvee \Phi_{Z_2}$. WLOG, there exists $z \in Z_1 \setminus Z_2$. Thus $\varphi_z \in \Phi_{Z_1}$, and as we've shown, $z \vDash \varphi_z$, and as shown in Question 2, this means that $z \vDash \bigvee \Phi_{Z_1}$. However, we have shown that if $z \neq z'$ with respect to the first k variables, then $z \nvDash \varphi_{z'}$. By construction, every $\varphi \in \Phi_{Z_2}$ is of such form $\varphi_{z'}$, that is, with $z' \neq z$, thus there is no formula in Φ_{Z_2} which z satisfies, and again, as shown in Question 2, this means that $z \nvDash \Phi_{Z_2}$. We have shown an assignment that satisfies $\bigvee \Phi_{Z_1}$ and not $\bigvee \Phi_{Z_2}$, thus the two formulae are not logically equivelant. □

5.3.3. *Part iii.* As we have exactly one formula for each set of assignments in $\wp(ASS(k))$, and they are all distinct (we have shown that they are not logically equivelant, thus they are also privately not equal as strings), then $|\Sigma_M| = |\wp(ASS(k)) = 2^{|ASS(k)}|$. By combinatorical considerations, $|ASS(k)|$ is the number of binary vectors of length k, that is, 2^k. Thus $|\Sigma_M| = 2^{2^k}$.

5.3.4. *part iv.*

Proof. Let Σ be a set of pairwise inequivelant formulae. We will show a 1-1 function from it to Σ_M, thus $|\Sigma| \leq |\Sigma_M|$.

Let $Ass_k(\varphi) = \{z \in ASS(k) | z \Vdash \varphi\}$. Then consider the following function:

$$f : \Sigma \to \Sigma_M, f(\varphi) = \bigvee \Phi_{Ass_k(\varphi)}$$

To show that it is 1-1, take $\varphi_1 \neq \varphi_2 \in \Sigma$. By the assumption, Σ formulae are pairwise inequivelant, thus $Ass_k(\varphi_1) \neq Ass_k(\varphi_2)$, thus, as we have shown, $\bigvee \Phi_{Ass_k(\varphi_1)} \not\sim \bigvee \Phi_{Ass_k(\varphi_2)}$, and privately, they are different as strings. $\square$

6. Question 6

6.1. **Part A.** The claim is false. Take $\Sigma_1 = \{p_0\}, \Sigma_2 = \{p_1\}$. Clearly, $\Sigma \vDash p_0 \wedge p_1$, because any assignment which satisfies Σ would have to satisfy p_0, p_1, and by $TT_\wedge$, this means it satisfies $p_0 \wedge p_1$. However, the assignemnt χ_{Σ_1}[2] satisfies Σ_1, but does not satisfy $p_0 \wedge p_1$ as it gives p_1 0, and similarily, χ_{Σ_2} satisfies Σ_2 but not $p_0 \wedge p_1$. Thus $\Sigma_1 \cup \Sigma_2$ is not partitioned into Σ_1, Σ_2.

6.2. **Part B.** The claim is false. Take $\Sigma = \{p_i | i \in \mathbb{N}\}$. Assume by contrast that Σ is partitioned into Σ_1, Σ_2. By definition, they are nonempty. WLOG, assume $p_1 \in \Sigma_1, p_2 \in \Sigma_2$. Again by definition, they are disjoint, thus $p_1 \notin \Sigma_2, p_2 \notin \Sigma_1$. $\Sigma \vDash p_1 \wedge p_2$, because as we have shown in class, only $z_{\mathbf{T}}$ satisfies Σ, thus p_1, p_2 are assigned 1. However, $\Sigma_1 \nvDash p_1 \wedge p_2$, because χ_{Σ_1} assigns 0 to p_2, and thus, while satisfying Σ_1, does not satisfy $p_1 \wedge p_2$, and similarily, χ_{Σ_2} satisfies Σ_2 but not $p_1 \wedge p_2$. Thus Σ is not partitioned into Σ_1, Σ_2. Our only contrast-assumption was that such Σ_1, Σ_2 exist that Σ is partitioned into them, therefore they do not.

6.3. **Part C.**

Lemma 2 (The theoretic theory theory)**.** *For any* $\Sigma \in \wp(\mathbf{WFF})$*,* $Con(\Sigma)$ *is a theory.*

Proof of Lemma **??**. Assume $Con(\Sigma) \vDash \alpha$. We want to show that $\Sigma \vDash \alpha$, and then $\alpha \in Con(\Sigma)$, thus $Con(\Sigma)$ is a theory. But if z satisfies Σ, then by definition of $Con(\Sigma)$ (as the set of formulae which are satisfied by all assignments which satisfy Σ), z satisfies $Con(\Sigma)$. As per the assumption, now we have that z satisfies α, and we have shown that $\Sigma \vDash \alpha$. $\square$

Proof of 6C. Assume Σ is partitioned into Σ_1, Σ_2. Then by definition of a partition, $Con(\Sigma) = Con(\Sigma_1) \cup Con(\Sigma_2)$. Thus by Lemma **??**, $Con(\Sigma), Con(\Sigma_1), Con(\Sigma_2)$ are theories, and from the equality, so is $Con(\Sigma_1) \cup Con(\Sigma_2)$. But by **??**, this means that either $Con(\Sigma_1) \subseteq Con(\Sigma_2)$ or vice versa. Assume the former, then if $\Sigma_1 \vDash \alpha$, then $\Sigma_2 = \Sigma \setminus \Sigma_1 \vDash \alpha$, and Σ_1 is redundant. Identically, assuming the latter gives that Σ_2 is redundant. $\square$

[2] As per usual, $\chi_A(t) = \begin{cases} 1, & t \in A \\ 0, & t \notin A \end{cases}$

LOGIC & SET THEORY - HW 8

OHAD LUTZKY

Please return to cell 7

1. Question 1

1.1. Part A.

Proof. We wish to show that $\{\to, \heartsuit\}$ is functionally complete. It will suffice to show that every formula $\varphi \in \mathbf{WFF}_{\{\to,\mathbf{F}\}}$ can be converted to a logically equivelant formula $\varphi' \in \mathbf{WFF}_{\{\to,\heartsuit\}}$, as we have seen in class that $\mathbf{WFF}_{\{\to,\mathbf{F}\}}$is functionally complete. We will show this by induction on $\mathbf{WFF}_{\{\to,\mathbf{F}\}}$.

Basis: For $\varphi = p_i$, φ is already in $\mathbf{WFF}_{\{\to,\heartsuit\}}$ without conversion, and they are trivially logically equivelant.

For $\varphi = \mathbf{F}$, take $\varphi' = \heartsuit p_0$. By $TT_\heartsuit$, $M(\varphi', z)$ is 0 for any assignment z, as is $M(\varphi, 0)$, so the two are equivelant.

For $\varphi = \mathbf{T}$, take $\varphi' = (\heartsuit p_0 \to \heartsuit p_0)$.

$$\begin{aligned} M(\varphi', z) &= M((\heartsuit p_0 \to \heartsuit p_0), z) \\ &= TT_\to(M(\heartsuit p_0, z), M(\heartsuit p_0, z)) \\ &= TT_\to(0, 0) = 1 \end{aligned}$$

Closure: Assume the claim holds for φ_1, φ_2, that is, they are logically equivelant to $\varphi'_1, \varphi'_2 \in \mathbf{WFF}_{\{\to,\heartsuit\}}$. Consider $\varphi = \varphi_1 \to \varphi_2$, and $\varphi' = \varphi'_1 \to \varphi'_2$. Clearly $\varphi' \in \mathbf{WFF}_{\{\to,\heartsuit\}}$. As for logical equivelance,

$$\begin{aligned} M(\varphi', z) &= M(\varphi'_1 \to \varphi'_2, z) \\ &= TT_\to(M(\varphi'_1, z), M(\varphi'_2, z)) \end{aligned}$$

But by the inductive assumption,

$$\begin{aligned} &= TT_\to(M(\varphi_1, z), M(\varphi_2, z)) \\ &= M(\varphi, z) \end{aligned}$$

□

1.2. Part B.

Proof. We wish to show that $\{\to, \oplus\}$ is functionally complete. It will suffice to show that every formula $\varphi \in \mathbf{WFF}_{\{\to,\mathbf{F}\}}$ can be converted to a logically equivelant formula $\varphi' \in \mathbf{WFF}_{\{\to,\oplus\}}$, as we have seen in class that $\mathbf{WFF}_{\{\to,\mathbf{F}\}}$is functionally complete. We will show this by induction on $\mathbf{WFF}_{\{\to,\mathbf{F}\}}$.

Basis: For $\varphi = p_i$, φ is already in $\mathbf{WFF}_{\{\to,\oplus\}}$ without conversion, and they are trivially logically equivelant.

For $\varphi = \mathbf{F}$, take $\varphi' = (p_0 \oplus p_0)$. By $TT_\oplus$, $M(\varphi', z)$ is 0 for any assignment z, as is $M(\varphi, 0)$, so the two are equivelant.

For $\varphi = \mathbf{T}$, take $\varphi' = (p_0 \oplus p_0) \to (p_0 \oplus p_0)$. Similarily to Part A, again we have that φ, φ' are logically equivlenat.

Closure: Precisely identical to Part A. Save the trees!

□

3. Question 3

3.1. **Part A.** The claim is true.

Proof. We are asked to show that $\{\psi \to \alpha, \alpha \to \beta, \beta \to \varphi\} \vdash \psi \to \varphi$. By deduction, it is enough to show that $\{\psi, \psi \to \alpha, \alpha \to \beta, \beta \to \varphi\} \vdash \varphi$. The following proof sequence will show that:

1.	ψ	Assumption
2.	$\psi \to \alpha$	Assumption
3.	$\alpha \to \beta$	Assumption
4.	$\beta \to \varphi$	Assumption
5.	α	MP(1,2)
6.	β	MP(5,3)
7.	φ	MP(4,6)

Thus $\{\psi, \psi \to \alpha, \alpha \to \beta, \beta \to \varphi\} \vdash \varphi$. □

3.2. **Part B.** The claim is true.

We will prove a stronger property, that given the same conditions, for all $\varphi \in \mathbf{WFF}_{\{\to,\mathbf{F}\}}$ it holds both that $\varphi \vdash \mathit{subst}(\varphi, s)$ and $\mathit{subst}(\varphi, s) \vdash \varphi$.

Proof. We'll prove by induction on the structure of *subst*.

Basis: If $\varphi = p_i$, then $\mathit{subst}(\varphi, s) = s(p_i)$. We are given that $p_i \vdash s(p_i)$, thus $\varphi \vdash \mathit{subst}(\varphi, s)$. Similarily, we are given that $s(p_i) \vdash p_i$, thus $\mathit{subst}(\varphi, s) \vdash \varphi$.

If $\varphi = \mathbf{F}$, then $\mathit{subst}(\varphi, s) = \mathbf{F}$, then since clearly $\mathbf{F} \vdash \mathbf{F}$ (a proof sequence of length 1), we have that $\varphi \vdash \mathit{subst}(\varphi, s)$ and $\mathit{subst}(\varphi, s) \vdash \varphi$.

Closure: Assume that for $\varphi_1, \varphi_2 \in \mathbf{WFF}_{\{\to,\mathbf{F}\}}$, it holds that $\mathit{subst}(\varphi_1, s) \vdash \varphi_1, \varphi_1 \vdash \mathit{subst}(\varphi_1, s), \mathit{subst}(\varphi_2, s) \vdash \varphi_2, \varphi_2 \vdash \mathit{subst}(\varphi_2, s)$. We need to show that $\mathit{subst}(\varphi_1 \to \varphi_2, s) \vdash \varphi_1 \to \varphi_2, \varphi_1 \to \varphi_2 \vdash \mathit{subst}(\varphi_1 \to \varphi_2, s)$. Note that $\mathit{subst}(\varphi_1 \to \varphi_2, s) = \mathit{subst}(\varphi_1, s) \to \mathit{subst}(\varphi_2, s)$. By the deduction theorem, it suffices to show that $\{\varphi_1 \to \varphi_2, \mathit{subst}(\varphi_1, s)\} \vdash \mathit{subst}(\varphi_2, s)$, and $\{\varphi_1, \mathit{subst}(\varphi_1, s) \to \mathit{subst}(\varphi_2, s)\} \to \varphi_2$.

	1.	$\mathit{subst}(\varphi_1, s)$	(Assumption)
	...	$[\mathit{subst}(\varphi_1, s) \vdash \varphi_1]$	
	n.	φ_1	
First claim:	$n+1$.	$\varphi_1 \to \varphi_2$	(Assumption) [1]
	$n+2$.	φ_2	(MP($n, n+1$))
	....	$[\varphi_2 \vdash \mathit{subst}(\varphi_2, s)]$	
	m.	$\mathit{subst}(\varphi_2, s)$	

	1.	φ_1	(Assumption)
	...	$[\varphi_1 \vdash \mathit{subst}(\varphi_1, s)]$	
	n.	$\mathit{subst}(\varphi_1, s)$	
Second claim:	$n+1$.	$\mathit{subst}(\varphi_1, s) \to \mathit{subst}(vp_2, s)$	(Assumption)
	$n+2$.	$\mathit{subst}(\varphi_2, s)$	(MP($n, n+1$))
	....	$[\mathit{subst}(\varphi_2, s) \vdash \varphi_2]$	
	m.	φ_2	

□

[1] I denote by $[\psi \vdash \varphi]$ or $[\Sigma \vdash \varphi]$ that here one inserts the proof sequence that relies only on ψ or Σ respectedly, and ends with φ (without the last step, which is inserted explicitly). Naturally, it is only valid if we have indeed listed ψ or all of Σ before this point in the proof, and the stated condition does indeed hold. If there is a more widely accepted form of notation for this, please let me know.

3.3. **Part C.** The claim is true.

Proof. We are given a substitution s such that for any $i \in \mathbb{N}$, both $p_i \vdash s(p_i)$ and $s(p_i) \vdash p_i$. Therefore, by **??**, we have that for any $\psi \in \mathbf{WFF}_{\{\to,\mathbf{F}\}}$, both $\psi \vdash subst(\psi, s)$ and $subst(\psi, s) \vdash \psi$. Privately, this also holds for any $\psi \in \Sigma$. Since $\Sigma \vdash \varphi$, and all proof sequences are finite, we know that only a finite number of formulae from Σ can be used in a proof. Therefore there exists a finite set $\Sigma' \subseteq \Sigma$ such that $\Sigma' = \{\sigma_0, \sigma_1, \dots, \sigma_n\} \vdash \varphi$. We can therefore construct the following proof sequence to show $subst(\Sigma', s) \vdash subst(\varphi, s)$, and by monotonicity, we will have that $subst(\Sigma, s) \vdash subst(\varphi, s)$.

1.	$subst(\sigma_0, s)$	(Assumption)
...		
n.	$subst(\sigma_n, s)$	(Assumption)
...	$[subst(\sigma_i, s) \vdash \sigma_i]$	
m.	σ_0	
...		
$m+n$.	σ_n	
...	$[\Sigma' \vdash \varphi]$	
ξ.	φ	
...	$[\varphi \vdash subst(\varphi, s)]$	
ζ.	$subst(\varphi, s)$	

□

4. Question 4

4.1. **Part A.**

Proof. We'll prove by induction on $Ded_N(\emptyset)$.

Basis: There are no assumptions, so it suffices to show that the axioms are tautologies.

If $\varphi = \neg\alpha \to (\alpha \to \neg\alpha)$, for some $\alpha \in \mathbf{WFF}_{\{\neg,\to\}}$ then for any assignment z,

$$M(\neg\alpha \to (\alpha \to \neg\alpha), z) = TT_\to(TT_\neg(M(\alpha, z)), TT_\to(M(\alpha, z), TT_\neg(M(\alpha, z))))$$

Now, $M(\alpha, z)$ is some constant $m \in \{0, 1\}$. But for any such constant, clearly this expression evaluates to 1:

- For $m = 0$,
$$\dots = TT_\to(TT_\neg(0), TT_\to(0, TT_\neg(0))) = TT_\to(1, 1) = 1$$
- For $m = 1$,
$$\dots = TT_\to(TT_\neg(1), TT_\to(1, TT_\neg(1))) = TT_\to(0, TT_\to(1, 0)) = TT_\to(0, 0) = 1$$

If $\varphi = (\alpha \to (\alpha \to \neg\alpha)) \to (\alpha \to \neg\alpha)$, then for any assignment z,

$$M(\alpha \to (\alpha \to \neg\alpha)) \to (\alpha \to \neg\alpha), z) =$$
$$= TT_\to(TT_\to(M(\alpha, z), TT_\neg(M(\alpha, z))), TT_\to(M(\alpha, z), TT_\neg(M(\alpha, z))))$$

Now, $M(\alpha, z)$ is some constant $m \in \{0, 1\}$. But for any such constant, this expression evaluates to 1:

- For $m = 0$,
$$\begin{aligned} \dots &= TT_\to(TT_\to(0, TT_\neg(0)), TT_\to(0, TT_\neg(0))) \\ &= TT_\to(TT_\to(0, 1), TT_\to(0, 1)) \\ &= TT_\to(1, 1) = 1 \end{aligned}$$

- For $m = 1$,

$$\begin{aligned}\ldots &= TT_{\rightarrow}(TT_{\rightarrow}(1, TT_{\neg}(1)), TT_{\rightarrow}(1, TT_{\neg}(1))) \\ &= TT_{\rightarrow}(TT_{\rightarrow}(1,0), TT_{\rightarrow}(1,0)) \\ &= TT_{\rightarrow}(0,0) = 1\end{aligned}$$

Closure: Assume that $\varphi \rightarrow \psi, \varphi \in Ded_N(\emptyset)$ are tautolgies, then $M(\varphi \rightarrow \psi, z) = 1$, for any assignment z. However,

$$\begin{aligned}1 &= M(\varphi \rightarrow \psi, z) \\ &= TT_{\rightarrow}(M(\varphi, z), M(\psi, z))\end{aligned}$$

But seeing as φ is a tautology as well,

$$= TT_{\rightarrow}(1, M(\psi, z))$$

And this can only hold if $M(\psi, z) = 1$. We made no assumptions on z, thus it must hold for any assignment z, and we have that ψ is a tautology.

□

4.2. Part B.

Proof. We'll prove by induction on $Ded_N(\emptyset)$.

Basis: If $\varphi = \neg\alpha \rightarrow (\alpha \rightarrow \neg\alpha)$ for some $\alpha \in \mathbf{WFF}_{\{\neg,\rightarrow\}}$, then $\varphi^* = \alpha \rightarrow (\alpha \rightarrow \alpha)$. For any assignment z, $M(\alpha, z)$ can be either 0 or 1. If $M(\alpha, z) = 1$, then $M(\varphi^*, z) = TT_{\rightarrow}(1, TT_{\rightarrow}(1,1)) = 1$, and if $M(\alpha, z) = 0$, then $M(\varphi^*, z) = TT_{\rightarrow}(0, TT_{\rightarrow}(0,0)) = 1$, thus $\models \varphi^*$.

If $\varphi = (\alpha \rightarrow (\alpha \rightarrow \neg\alpha)) \rightarrow (\alpha \rightarrow \neg\alpha)$, then $\varphi^* = (\alpha \rightarrow (\alpha \rightarrow \alpha)) \rightarrow (\alpha \rightarrow \alpha)$. Therefore,

$$M(\varphi^*, z) = TT_{\rightarrow}(TT_{\rightarrow}(M(\alpha, z), TT_{\rightarrow}(M(\alpha, z), M(\alpha, z))), TT_{\rightarrow}(M(\alpha, z), M(\alpha, z)))$$

For any assignment z, either $M(\alpha, z) = 1$, in which case

$$\cdots = TT_{\rightarrow}(TT_{\rightarrow}(1, TT_{\rightarrow}(1,1)), TT_{\rightarrow}(1,1)) = 1$$

…or $M(\alpha, z) = 0$, in which case

$$\begin{aligned}\ldots &= TT_{\rightarrow}(TT_{\rightarrow}(0, TT_{\rightarrow}(0,0)), TT_{\rightarrow}(0,0)) \\ &= TT_{\rightarrow}(TT_{\rightarrow}(0,1), 1) \\ &= TT_{\rightarrow}(1,1) = 1\end{aligned}$$

And again, we have that $\models \varphi^*$.

Closure: Assume $\varphi, \varphi \rightarrow \psi \in Ded_N(\emptyset)$ and $\models \varphi^*, (\varphi \rightarrow \psi)^*$. By definition of *, this also means that $\models \varphi^* \rightarrow \psi^*$, so we have that for any assignment z,

$$\begin{aligned}1 &= M(\varphi^* \rightarrow \psi^*, z) = TT_{\rightarrow}(M(\varphi^*, z), M(\psi^*, z)) \\ &= TT_{\rightarrow}(1, M(\psi^*, z))\end{aligned}$$

And again, this is only possible if $\models \psi^*$.

□

4.3. **Part C.**

Disproof. Take $\varphi = \neg(p_0 \to p_0) \to p_0$. Only two assignments are relevant - one which gives p_0 0, and one which gives it 1. In either case, the meaning function on φ will give 1, thus φ is a tautology. Assume by constrast that $\vdash_N \varphi$, then by **??**, we have that $\vDash \varphi^*$. But $\varphi^* = (p_0 \to p_0) \to p_0$, which is not a tautology - for $z_{\mathbf{T}}$, $M(\varphi^*, z_{\mathbf{F}}) = TT_{\to}(TT_{\to}(0,0),0) = 0$, and we have a contradiction. Thus the claim is false. □

5. Question 5

Proof. We will prove by structure induction on $Ded_{M_1}(\emptyset)$ that if $\alpha \in Ded_{M_1}(\emptyset)$, then α is not a contradiction.

Basis: If $\alpha = \neg p_i$, then clearly α is not a contradiction — $M(\alpha, z_{\mathbf{F}}) = 1$.
If $\alpha = (p_i \to p_j$, then α is not a contradiction — $M(\alpha, z_{\mathbf{T}}) = 1$.
If $\alpha = (\beta \to \beta)$, then as we've seen in class, α is a tautology, and privately not a contradiction.

Closure: If $\alpha_1, \alpha_2 \in Ded_{M_1}(\emptyset)$ are not contradictions, then there exists an assignment z for which $M(\alpha_1, z) = 1$. For this assignment,

$$\begin{aligned} M(\neg\alpha_1 \to \alpha_2, z) &= TT_{\to}(TT_{\neg}(M(\alpha_1, z)), M(\alpha_2, z)) \\ &= TT_{\to}(TT_{\neg}(1), M(\alpha_2, z)) \\ &= TT_{\to}(0, M(\alpha_2, z)) = 1 \end{aligned}$$

And thus $\neg\alpha_1 \to \alpha_2$ is not a contradiction.

□

LOGIC & SET THEORY — HW 9

OHAD LUTZKY

Please return to cell 7

1. Problem 1

1.1. **Part A.** Take $\delta_{(\gamma_1 \vee \gamma_2)} = ((\gamma_1 \to \mathbf{F}) \to \gamma_2)$.

γ_1	γ_2	$\gamma_1 \vee \gamma_2$	$(\gamma_1 \to \mathbf{F})$	$((\gamma_1 \to \mathbf{F}) \to \gamma_2)$
0	0	0	1	0
0	1	1	1	1
1	0	1	0	1
1	1	1	1	1

We have that $\delta_{(\gamma_1 \vee \gamma_2)}$ is logically equivalent to $\gamma_1 \vee \gamma_2$.

1.2. **Part B.** There are 3 claims here:

- A. X is maximally consistent
- B1. For all $\gamma_1 \in \Gamma_1, \gamma_2 \in \Gamma_2$, $X \vdash \delta_{(\gamma_1 \vee \gamma_2)}$.
- B2. For all $\gamma_1 \in \Gamma_1$, if $X \nvdash \gamma_1$, then for all $\gamma_2 \in \Gamma_2$, $X \vdash \gamma_2$.

Proof. First direction — assume A,B1, and we'll show B2.

Let $\gamma_1 \in \Gamma_1$ be a formula such that $X \nvdash \gamma_1$, and select an arbitrary $\gamma_2 \in \Gamma_2$. X is maximally consistent, thus $X \vdash \neg\gamma_1$. By soundness, we have that $X \vDash \neg\gamma_1$, and by completeness and B1, $X \vDash \delta_{(\gamma_1 \vee \gamma_2)}$. By Part A, we have that $TT_\vee = TT_{\delta_\vee}$, and by $TT_\vee$, we have that $X \vDash \gamma_2$. By completeness, $X \vdash \gamma_2$.

Second direction — assume A,B2, and we'll show B1.

Let $\gamma_1 \in \Gamma_1, \gamma_2 \in \Gamma_2$.

- If $X \vdash \gamma_1$, then by soundness, $X \vDash \gamma_1$, and by $TT_{\delta_\vee}$, $X \vDash \delta_{(\gamma_1 \vee \gamma_2)}$. By completeness, $X \vdash \delta_{(\gamma_1 \vee \gamma_2)}$.
- If $X \nvdash \gamma_1$, then by B2, $X \vdash \gamma_2$, and by soundness, $X \vDash \gamma_2$. By $TT_{\delta_\vee}$, $X \vdash \delta_{(\gamma_1 \vee \gamma_2)}$, and by completeness, $X \vdash \delta_{(\gamma_1 \vee \gamma_2)}$.

Third direction — assume B, and we'll show A.

Assume by contrast that A is false. We are given that X is consistent, so assuming that A is false means assuming that it is not maximal, and thus there are two different assignments z, z' which satisfy X. They are different, thus there is some p_i such that $z(p_i) \neq z'(p_i)$. B is supposed to hold for any Γ_1, Γ_2, so we'll take $\Gamma_1 = \{p_i\}, \Gamma_2 = \{\neg p_i\}$. The prefix of B holds: The only choice for γ_1, γ_2 is $p_i, \neg p_i$, and then $\delta_{(\gamma_1 \vee \gamma_2)}$is a tautology. However, the suffix of B does not hold. Both z, z' satisfy X, but one of them does not satisfy $\gamma_1 = p_0$. Thus $X \nvDash \gamma_1$. By B and completeness, this means that $X \vdash \gamma_2$, and by soundness $X \vDash \gamma_2$. But again, both z, z' satisfy X, and one of them does not satisfy $\gamma_2 = \neg p_0$, and thus $X \nvDash \gamma_2$ — a contradiction.

□

2. Problem 2

2.1. **Part A.** The claim is false. Take $\Sigma = \{\mathbf{F}\}$, $\alpha = p_0, \beta = p_1$. As we've shown in class, for any $\varphi \in \mathbf{WFF}$, $\{\mathbf{F}\} \vdash \varphi$, therefore $\Sigma \vdash \alpha, \beta$. However, $\alpha \nvDash \beta$, and by soundness $\alpha \nvdash \beta$, and this is true the other way around WLOG.

Claim 1 (Tautologies for tots). *All formulae $\varphi \in Ded_N(\Sigma)$ are tautologies, regardless of Σ.*

Proof of Claim **??**. We'll prove by structural induction.

Basis: In the basis of Ded_N we have axioms and assumptions. For axioms, we have already shown in class that our chosen axioms are tautologies. For assumptions, all assumptions are of the form $\alpha \to (p_0 \to p_0)$. By $TT_\to$, $p_0 \to p_0$ is a tautology, and again by $TT_\to$, $\alpha \to (p_0 \to p_0)$ is a tautology, regardless of α.

Closure:
- MP: Assume $\psi, \psi \to \varphi$ are tautologies. Then by $TT_\to$, since $M(\psi, z)$ is 1 for any z, and $M(\psi \to \varphi, z)$ is 1 for any z, it must hold that $M(\varphi, z)$ is 1 for all z, thus φ is a tautology.
- As we've shown in the basis, f_i is always a tautology.
- Assume α is a tautology. If α is not of the required form, then $g(\alpha) = \alpha$ is a tautology. Otherwise, Changing the index p_i to p_{i+1} still leaves α a tautology.

□

2.2. **Part B.** The claim is true, since we've shown that for any $\varphi \in Ded_N(\sigma)$, by Claim **??**, $\vDash \varphi$, and by monotonicity, $\Sigma \vDash \varphi$.

2.3. **Part C.** The claim is true, because given that for some Σ, $\Sigma \vdash_N \varphi$, we have shown that φ is a tautology, that is, $\vDash \varphi$. So by monotonicity we have that $\{\alpha\} \vDash \beta, \{\beta\} \vDash \alpha$.

2.4. **Part D.** The claim is false. Take $\Sigma = p_0$. Clearly, $\Sigma \vDash p_0$. However, p_0 is not a tautology ($z_{\mathbf{F}}$ does not satisfy it), and therefore $\Sigma \nvdash_N p_0$.

3. Problem 3

3.1. **Part A.** The claim is false. Take $A_1 = \{z \in ASS | z(p_0) = 0\}, A_2 = ASS \setminus A_1$. Clearly $ASS = A_1 \cup A_2$. However, ASS is not informative — if $\varphi \in \Gamma_{ASS}$, then any assignment satisfies it, and it is a tautology. All that remains is to show that A_1, A_0 are informative. A_1 is informative because $\neg p_0 \in \Gamma_{A_1}$ — any assignment which assigns 0 to p_0 satisfies $\neg p_0$. Similarily, $p_0 \in \Gamma_{A_2}$, because no assignment in A_2 assigns 0 to p_0.

3.2. **Part B.** The claim is false. Take A to be the set of all assignments which assign 1 to a finite number of variables. Take any finite subset $D \subseteq A$, then since any assignment $z \in D$ only assigns 1 to a finite number of variables, each one of them has a first variable to which it assigns 0, and from that point on only 0s are assigned. Therefore there is a variable p_i for which any $z \in D$ assigns $z(p_i) = 0$, and we have that $\neg p_i \in \Gamma_D$, and seeing as $\neg p_i$ is not a tautology, D is informative.

All that remains is to show that A is not informative. Assume $\varphi \in \Gamma_A$. φ is satisfied by any assignment which assigns 1 to a finite number of variables. Assume by negation that φ is, nevertheless, not a tautology. Then there exists some assignment z which does not satisfy it. Thus there is an assignment $z' \in A$ which identifies with z on any variable which appears in φ — this is possible because φ can only have a finite number of variables in it. And then we have that z' does not satisfy φ either, a contradiction. Then z is a tautology, and $\Gamma_A \subseteq TAUT$, and A is not informative.

3.3. Part C. The claim is true.

Proof. First direction:

$|A| = 1$, that is, $A = \{z\}$. Therefore $z \vDash \Gamma_A$, and it is satisfiable. Assume that $\Gamma_A \subsetneq X$, and X is satisfiable. Then X is satisfied by some assignment $z' \neq z$. Since those assignments are different, then there exists p_i such that $z(p_i) \neq z'(p_i)$.

- If $z(p_i) = 0$, then $z \vDash \neg p_i$, and $\neg p_i \in \Gamma_A$. However, $z' \nvDash \neg p_i$, and since $z' \vDash X$, $\neg p_i \notin X$, in contradiction to $\Gamma_A \subseteq X$.
- If $z(p_i) = 1$, then $z \vDash p_i$, and $p_i \in \Gamma_A$. However, $z' \nvDash p_i$, and since $z' \vDash X$, $p_i \notin X$, in contradiction to $\Gamma_A \subseteq X$.

Either way, we have a contradiction. Thus such a set X does not exist.

Second direction:

Assume by negation that $|A| \neq 1$. If $|A| = 0$ then $\Gamma_A = \mathbf{WFF}$, and since $\mathbf{F} \in \mathbf{WFF}$, Γ_A is not satisfiable, a contradiction. Then $|A| \geq 2$. Then there are $z_1, z_2 \in A$. Take $X = \Gamma_{\{z_1\}}$, then since $\{z_1\} \subseteq A$, then by definition of $\Gamma_\circ$, $\Gamma_A \subseteq \Gamma_{\{z_1\}} = X$. However, z_1 and z_2 disagree on some variable p_i. Assume WLOG that $z_1(p_i) = 1 \neq z_2(p_i)$, then $p_i \in X \setminus \Gamma_A$. Then $\Gamma_A \subsetneq X$, yet X is satisfiable — $z_1 \vDash X$, a contradiction. □

4. Problem 4

4.1. Part A.

$$((\alpha \to \beta) \to ((\beta \to \alpha) \to \mathbf{F})) \to \mathbf{F}$$

4.2. Part B.

Proof. Assume $(\alpha, \beta) \in R_\Sigma$. Then $\Sigma \vdash \varphi_{\alpha,\beta}$. By soundness, $\Sigma \vDash \varphi_{\alpha,\beta}$. As we were asked not to prove, $TT_{\varphi_{\circ,\circ}} = TT_\leftrightarrow$, thus any assignment which satisfies $\varphi_{\alpha,\beta}$, by $TT_{\varphi_{\circ,\circ}}$, satisfies $\varphi_{\beta,\alpha}$. Then $\Sigma \vDash \varphi_{\beta,\alpha}$, and by completeness, $\Sigma \vdash \varphi_{\beta,\alpha}$, and $(\beta, \alpha) \in R_\Sigma$. □

4.3. Part C. $|\mathbf{WFF}_{\{\to,\mathbf{F}\}}/R_\Sigma| = 1$

Proof. Let Σ be an inconsistent set. Thus any formula $\varphi \in \mathbf{WFF}$ can be proven by it — that is, $\Sigma \vdash \varphi$. In particular, this also holds true for any $\varphi_{\alpha,\beta}$, for any two formulae $\alpha, \beta \in \mathbf{WFF}$. Thus all formulae are equivalent under R_Σ, and there is only one equivalence class. □

4.4. Part D. $|\mathbf{WFF}_{\{\to,\mathbf{F}\}}/R_\Sigma| = 2$

Proof. Let Σ be a maximally consistent set. As we've shown in class, this means that there is precisely one assignment z such that $z \vDash \Sigma$. Take two formulae $\alpha, \beta \in \mathbf{WFF}$. Iff $M(\alpha, z) = M(\beta, z)$, then by $TT_\leftrightarrow$, $M(\varphi_{\alpha,\beta}, z) = 1$, and since z is the only assignment which satisfies Σ, $\Sigma \vdash \varphi_{\alpha,\beta}$, and by completeness, $(\alpha, \beta) \in R_\Sigma$. Therefore any formula φ is equivalent under R_Σ precisely to any formula ψ which receives $M(\psi, z) = M(\varphi, z)$, and seeing as there are two options for this value (1 or 0), then there are two equivalence classes. □

5. Problem 5

5.1. Part A.

Proof. First direction:

Assume $K \neq \emptyset$. Let Σ be a set of formulae such that Σ is sound for K. Assume by contrast that Σ is inconsistent, then $\Sigma \vdash \mathbf{F}$. Σ is sound for K, thus $\mathbf{F} \in Th(K)$. Therefore, for any assignment $z \in K$, $z \vDash \mathbf{F}$. But there do not exist any assignments which satisfy $\mathbf{F}$, thus $K = \emptyset$ — a contradiction.

Second direction:

Assume by contrast that $K = \emptyset$. Then by definition, trivially, $Th(K) = \mathbf{WFF}$. Take $\Sigma = \mathbf{WFF}$. For any formula φ, $\mathbf{WFF} \vdash \varphi$ because $\mathbf{WFF}$ is inconsistent. Thus $\mathbf{WFF}$ is sound for K. But $\mathbf{F} \in \mathbf{WFF}$, thus $\Sigma = \mathbf{WFF}$ is not consistent — a contradiction. □

5.2. **Part B.**

Proof. First direction:

Assume $|K| \leq 1$, and let Σ be complete for K.

- If $K = \emptyset$, by Part A, $Th(K) = \mathbf{WFF}$. We now need to show that Σ is maximal. Take a formula φ. Then since $Th(K) = \mathbf{WFF}$, $\varphi \in Th(K)$. Σ is complete for K, thus $\Sigma \vdash \varphi$. We have shown that Σ is maximal.
- If $|K| = 1, K = \{z\}$. Let Σ be complete for K, and φ be an arbitrary formula.
  - If $z \vDash \varphi$, then $z \in Th(K)$. Since Σ is complete for K, $\Sigma \vdash \varphi$.
  - If $z \nvDash \varphi$, then by $TT_\neg$, $z \vDash \neg\varphi$. Thus $\neg\varphi \in Th(K)$. Σ is complete for K, therefore $\Sigma \vdash \neg\varphi$.

  We have shown that either $\Sigma \vdash \varphi$ or $\Sigma \vdash \neg\varphi$ for an arbitrary formula φ, thus Σ is maximal.

Second direction:

Assume by contrast $|K| > 1$. Choose $\Sigma = Th(K)$. Σ is complete for K - if $\varphi \in Th(K)$, then $\varphi \in \Sigma$, thus $\Sigma \vdash \varphi$ with a trivial proof sequence. For a contradiction, we will show that Σ is not maximal.

$|K| \geq 2$, thus $z_1, z_2 \in K, z_1 \neq z_2$. z_1, z_2 disagree on some variable p_i - either $z_1 \nvDash p_i$ or $z_2 \nvDash p_i$. Thus $\Sigma \nvDash p_i$, and by soundness, $\Sigma \nvdash p_i$. However, the same argument also shows that $\Sigma \nvDash \neg p_i$, and by soundness, $\Sigma \nvdash \neg p_i$. Thus Σ is not maximal, and we have our contradiction. □

5.3. **Part C.**

Proof. First direction:

Assume $|K| \geq 2$. $z_1, z_2 \in K$ disagree on some variable p_i. Thus, $p_i, \neg p_i \notin Th(K)$. Σ is sound for K, thus $\Sigma \nvdash p_i, \neg p_i$, and Σ is not maximal.

Second direction:

Assume $|K| \leq 1$, and choose $\Sigma = Th(K)$ - we will show it to be both sound for K and maximal. Let φ be a formula such that $\Sigma \vdash \varphi$. By soundness we have that $\Sigma \vDash \varphi, Th(K) \vDash \varphi$, and by definition of Th, $\varphi \in Th(K)$.

Now we will show that Σ is maximal.

- If $K = \emptyset$, then similarily to part A, $Th(K) = \mathbf{WFF}$, thus $Th(K) \vdash \varphi$ for any $\varphi \in \mathbf{WFF}$. Therefore $Th(K)$ is maximal.
- If $K = \{z\}$, then let φ be some formula.
  - If $z(\varphi) = 1$, then $\varphi \in Th(K)$, and $Th(K) \vdash \varphi$ trivially.
  - If $z(\varphi) = 0$, then $\neg\varphi \in Th(K)$, and $Th(K) \vdash \neg\varphi$ trivially.

  We have shown that either $Th(K) \vdash \varphi$ or $Th(K) \vdash \neg\varphi$, thus $Th(K)$ is maximal.

□

LOGIC & SET THEORY — HW 11

OHAD LUTZKY

2. Question 2

2.1. Part A. The statement is a tautology.

Proof. Let $\mathfrak{A} = \langle A, R^M, P^M, F^M \rangle$ be a τ-structure and z be an assignment. We will evaluate the meaning function $M(\varphi_1, \mathfrak{A}, z)$:

$$\begin{aligned} M(\varphi_1, \mathfrak{A}, z) &= M(\forall v_1 P(v_1) \to \forall v_2 P(F(v_2)), \mathfrak{A}, z) \\ &= TT_{\to}(M(\forall v_1 P(v_1), \mathfrak{A}, z), M(\forall v_2 P(F(v_2)), \mathfrak{A}, z)) \end{aligned}$$

To show that $TT_{\to}$ always receives 1, we will show that if $\mathfrak{A} \vDash_z \forall v_1 P(v_1)$, then $\mathfrak{A} \vDash_z \forall v_2 P(F(v_2))$. Assuming that indeed the prefix is satisfied, we see that for any $d \in A$, $\mathfrak{A} \vDash_{z[v_2 \leftarrow d]} P(v_2)$, which in turn means that for any $d \in A, d \in P^{\mathfrak{A}}$.

Note that $F^{\mathfrak{A}}$ is a function $A \to A$, thus for any $d \in D$, $F^{\mathfrak{A}}(d) \in P^{\mathfrak{A}}$. This means that for any assignment z', $\mathfrak{A} \vDash_z P(F(v_2)$. In particular, this also holds for corrected assignments, hence $\mathfrak{A} \vDash_z \forall v_2 P(F(v_2))$.

□

2.2. Part B. The statement is not a tautology. Consider

$$\mathfrak{A} = \langle A = \{0, 1\}, R^{\mathfrak{A}} = \emptyset, P^{\mathfrak{A}} = \{0\}, F^{\mathfrak{A}} = d \mapsto 0 \rangle, z(v_i) = 1$$

Under any assignment, particularly a corrected one, the prefix is satisfied — as $F^{\mathfrak{A}}(v_1) = 0$ for any value of v_1, $F^{\mathfrak{A}}(v_1) \in P^{\mathfrak{A}}$ for any assignment, and we have that $\mathfrak{A} \vDash_z \forall v_1 P(F(v_1))$. As for the suffix, however — its meaning evaluates to 0: There exists a value $d = 1 \in A$ for which $d \notin P^{\mathfrak{A}}$, thus it is not true that "for every $d \in A$, $\mathfrak{A} \vDash_{z[v_2 \leftarrow d]} P(v_2)$", and thus $\mathfrak{A} \nvDash_z \forall v_2 P(v_2)$. Due to the properties of $TT_{\to}$, this means that $\mathfrak{A} \nvDash_z \varphi_2$.

2.3. Part C. The statement is not a tautology. Take

$$\mathfrak{A} = \langle \mathbb{Z}, <, \emptyset, + \rangle, z(v_i) = 0$$

Under any assignment, the meaning of the prefix is true: For any integer a there exists an integer b such that $a < b$. Therefore, for any assignment z which assigns $z(v_1) = a$, there exists $b \in \mathbb{Z}$ such that $M(R(v_1, v_2), \mathfrak{A}, z[v_2 \leftarrow b]) = 1$. Hence for any such assignment z, $M(\exists v_2 R(v_1, v_2), \mathfrak{A}, z) = 1$. Equivalently, for any assignment z at all, for any $a \in \mathbb{Z}$, $M(\exists v_2 R(v_1, v_2), \mathfrak{A}, z[v_1 \leftarrow a]) = 1$, which means that $M(\forall v_1 \exists v_2 R(v_1, v_2), \mathfrak{A}, z) = 1$.

3. Question 3

3.1. Part A.

Proof. We will notate $M = \langle A, P^M, F^M, c^M \rangle$. Assume by contrast that there exists a term t over τ and an assignment z for which $M \nvDash_z P(t)$. Hence it does not hold that $\bar{z}(t) \in P^M$. Since by definition, $\bar{z}(t) \in A$, then we have found an assignment z and an element $d \in A$ for which $M \nvDash_{z[v_1 \leftarrow d]} P(v_1)$. Consequently, $M \nvDash \forall v_1 P(v_1)$.

□

3.2. **Part B.** The requested set is defined as an inductive set $X_{\tilde{B},\tilde{F}}$ with $B = \{c\}$, $\tilde{F} = \{t \mapsto F(t)\}$

3.3. **Part C.** We will show by structural induction over $X_{\tilde{B},\tilde{F}}$ as defined.

Basis: There is only one case in the basis, c. Let M, z be a τ-structure and an assignment respectively. If $M \vDash_z \Sigma$, then by definition $M \vDash_z P(c)$.

Closure: We will assume by induction that $\Sigma \vDash P(t)$, and show that $\Sigma \vDash P(F(t))$. Let M be a τ-structure and z be an assignment. We will denote $t^M = \bar{z}(t)$. If $M \vDash_z \Sigma$, then $M \vDash_z \forall v_1[P(v_1) \to P(F(v_1))]$. This holds only if for any $d \in A$ (A being the domain of the structure M), $M \vDash_{z[v_1 \leftarrow d]} P(v_1) \to P(F(v_1))$. In particular, it must hold for $d = t^M$. Note that for this choice of d, the prefix is satisfied: By the inductive assumption, $\Sigma \vDash P(t)$, thus $M \vDash_z P(t)$. This shows that

$$t^M = \bar{z}[v_1 \leftarrow t^M](v_1) \in P^M$$

Thus, $M \vDash_{z[v_1 \leftarrow t^M]} P(v_1)$. Due to the properties of $TT_{\to}$, we have that $M \vDash_{z[v_1 \leftarrow t^M]} P(F(v_1))$. Therefore, $\bar{z}[v_1 \leftarrow t^M](F(v_1)) \in P^M$. We note that

$$\bar{z}[v_1 \leftarrow t^M](F(v_1)) = \bar{z}(F(t))$$

Therefore, $\bar{z}(F(t)) \in P^M$ — and we have shown that $M \vDash_z P(F(t))$.

3.4. **Part D.** The claim is false. Take $\mathfrak{A} = \langle\{0,1\}, \{0\}, a \mapsto \{0\}, 0\rangle$. Under any assignment, both statements are satisfied - in the latter obviously $0 \in \{0\}$, and for any assignment to v_1, the suffix of the former is satisfied as $F^{\mathfrak{A}}(\dots) = 0 \in \{0\}$, and thus the entire statement is satisfied. However, the statement $\forall v_1 P(v_1)$ is not satisfied, as for $d = 1$, $\bar{z}[v_1 \leftarrow d](v_1) = 1 \notin \{0\}$, thus $\Sigma \nvDash \forall v_1 P(v_1)$.

4. Question 4

4.1. **Part A.** The claim is false. Consider $M = \langle\mathbb{Z}, \leq, +\rangle, M' = \langle\{0\}, \{0,0\}, +\rangle$. Clearly, $\{0\} \subseteq Z$, $\{0,0\} = \text{“}\leq\text{”} \cap \{0\}^2$, if $a, b \in \{0\}$ then $a + b = 0 \in \{0\}$, and $0 + 0 = 0$ in M as well. However, consider the term $F(v_1, v_1)$ specifies 0 in M' (for any assignment of v_1 within $\{0\}$, $\bar{z}_{M'}(F(v_1, v_1)) = 0 + 0 = 0$. However, in M, $F(v_1, v_1)$ does not specify 0. For example, with the assignment $z = v_i \mapsto 1$, $\bar{z}_M(F(v_i, v_i)) = 2$.

4.2. **Part B.** The claim is true.

Lemma 1. *If $v_1, \dots, v_n$ are the free variables of φ, $d_1, \dots, d_n \in B$, and $z(v_1) = d_1, \dots, z(v_n) = d_n$, then $M(\varphi, M', z) = M(\varphi, M, z)$.*

Proof of Lemma **??**. We will prove inductively that for any such z, and a term t with only the variables in $v_1, \dots, v_n$, $\bar{z}_M(t) = \bar{z}_{M'}(t)$, and as a result, $\bar{z}_M(t) \in B$.

Basis: If $t = v_i$ with $1 \leq i \leq n$, then $z_M(t) = z_{M'}(t)$ trivially.

Closure: If the claim holds for terms t_1, t_2, then by definition of $F^{M'}$, $z_M(t_i) = z_{M'}(t_i) \in B$. Then by definition of a substructure,

$$\begin{aligned}
\bar{z}_{M'}(F(t_1, t_2)) &= F^{M'}(\bar{z}_{M'}(t_1), \bar{z}_{M'}(t_2)\\
&= F^{M'}(\bar{z}_M(t_1), \bar{z}_M(t_2))\\
&= F^M(\bar{z}_M(t_1), \bar{z}_M(t_2))\\
&= z_M(F(t_1, t_2))
\end{aligned}$$

Now we will prove that for any such z and an atomic formula φ with only the variables in $v_1, \ldots, v_n$, $M(\varphi, M', z) = M(\varphi, M, z)$. For formulas of the form $t_1 \approx t_2$, this clearly holds because we've shown that $\bar{z}_M(t) = \bar{z}_{M'}(t)$. It remains to show for formulas of the form $R(t_1, t_2)$. $M(R(t_1, t_2), M, z) = 1$ iff $(\bar{z}_M(t_1), \bar{z}_M(t_2)) \in R^M)$. But as we've shown, for this kind of z, $\bar{z}_M(t_i) \in B$, thus this holds iff $(\bar{z}_M(t_1), \bar{z}_M(t_2)) \in R^M \cap B^2$. By definition of a substructure, $R^M \cap B^2 = R^{M'}$, so this holds iff $(\bar{z}_M(t_1), \bar{z}_M(t_2)) \in R^{M'}$, and by the equality we've shown, all of this holds iff $(\bar{z}_{M'}(t_1), \bar{z}_{M'}(t_2)) \in R^{M'}$, which is true iff $M(\varphi, M', z) = 1$.

We have shown that atomic formulae get the same meaning in both M and M' under our specified kind of assignment, and due to the properties of the inductive definition of FOL, all formulae get the same meaning in both M and M' under these assignments. □

Proof of Part B. First direction:

Consider $(d_1, \ldots, d_n) \in [\varphi]_{M'}$. By definition of a substructure, $D^{M'} \subseteq D^M$, thus $(d_1, \ldots, d_n) \in B^n$, and all that remains is to show $(d_1, \ldots, d_n) \in [\varphi]_M$. By definition of $[\varphi]_{M'}$, for any assignment z such that $z(v_1) = d_1, \ldots, z(v_n) = d_n$, $M' \vDash_z \varphi$. Then by Lemma **??**, $M \vDash_z \varphi$, thus $(d_1, \ldots, d_n) \in [\varphi]_M$.

Second direction:

Consider $(d_1, \ldots, d_n) \in [\varphi]_M \cap B^n$. By definition of $[\varphi]_M$, for any assignment z such that $z(v_1) = d_1, \ldots, z(v_n) = d_n$, $M \vDash_z \varphi$. Also, $(d_1, \ldots, d_n) \in B^n$. Then by Lemma **??**, $M' \vDash_z \varphi$, thus $(d_1, \ldots, d_n) \in [\varphi]_{M'}$. □

4.3. **Part C.** The claim is false. Consider M, M' as defined previously, and $\varphi = \forall v_2 R(v_1, v_2)$. Clearly, $[\varphi]_{M'} = \{0\}$, as the formula is satisfied by any assignment in M'. However, $[\varphi]_M = \emptyset$: $M \vDash_z \varphi$ iff $\bar{z}[v_2 \leftarrow d](v_1) < \bar{z}[v_2 \leftarrow d](v_2)$, or equivalently $z(v_1) < d$, for any d. We know that there is no such assignment on v_1, thus there is no $d_1 \in [\varphi]_M$.

5. Question 5

5.1. **Part A.**

Proof. Consider the atomic formula $(\varphi \to \varphi^f)$. Due to the properties of $TT_\to$, it will suffice to show that if $M \vDash \varphi$, then $M \vDash \varphi^f$. Since we are disregarding the equality symbol, then φ is of the form $P(t)$ for some term t. We know that φ is satisfied, therefore for any assignment z, $\bar{z}(t) \in P^M$. It remains to show that $\bar{z}(t^f) \in P^M$, t^f being the replacement of any x by $f(x)$ in t. We will prove this by structural induction over the terms:

Basis: Take $t = c$, and assume $z(c) \in P^M$. As $c^f = c$, (it has no variables), we have that $z(c^f) \in P^M$.

Take $t = v_i$, and let z be an assignment. Assume $z(v_i) \in P^M$. $t^f = f(v_i)$. By monotonicity, we have that $M \vDash \forall v_i((P(v_i) \to P(f(v_i)))$, meaning that for any $d \in D$, D being the domain of M, $M \vDash_{z[v_i \leftarrow d]} P(v_i) \to P(f(v_i))$. This must also hold for the uncorrected z, that is, $M \vDash_z P(v_i) \to P(f(v_i))$. Observing $TT_\to$, and noting that by our assumption $M \vDash_z P(v_i)$, we see that $M \vDash_z P(f(v_i))$. This is satisfied only if $\bar{z}(f(v_i)) \in P^M$.

Closure: Assume that for the term t, $\bar{z}(t) \in P^M$. By the exact same argument as in the basis, we have that $\bar{z}(f(t)) \in P^M$.

□

5.2. **Part B.**

Proof. We will show by structural induction over FOL.

Basis: We've shown in Part A that for atomic formulae, if $M \vDash \varphi$ then $M \vDash \varphi^f$, which suffices.

Closure: Assume that for ψ_1, ψ_2, if $M \vDash \psi_i$ then $M \vDash \psi_i^f$.

For the case of $\vee$, it suffices to show that if either $M \vDash \psi_1$ or $M \vDash \psi_2$, then either $M \vDash \psi_1^f$ or $M \vDash \psi_2^f$. Assume WLOG that $M \vDash \psi_1$. Then by the inductive assumption, $M \vDash \psi_1^f$.

For the case of $\wedge$, it suffices to show that if both $M \vDash \psi_1$ and $M \vDash \psi_2$, then $M \vDash \psi_1^f$ and $M \vDash \psi_2^f$ — but again, this is a direct consequence of our inductive assumption.

For the cases of the $\forall, \exists$ quantifiers — they have no effect. Our inductive assumption holds for all assignments, corrected or otherwise.

□