Non-time Dependent Fluctuations

Probability

$$w \propto \exp\left[-\frac{\Delta E - T\Delta S + P\Delta V}{kT}\right]\exp\left[-\frac{R_{min}}{kT}\right] = \exp\left[-\frac{\Delta T\Delta S - \Delta P\Delta V}{2kT}\right]$$

where R_{min} is minimal work to be done

Main quantities $\left\langle \Delta T^2 \right\rangle = \frac{kT^2}{C_V}$ $\left\langle \Delta V^2 \right\rangle = -kT\left(\frac{\partial V}{\partial P}\right)_{T,N}$

$$\left\langle \Delta S^2 \right\rangle = kC_P \quad \left\langle \Delta P^2 \right\rangle = -kT\left(\frac{\partial P}{\partial V}\right)_S$$

$$\left\langle \Delta E^2 \right\rangle = -kT\left[T\left(\frac{\partial P}{\partial T}\right)_V - P\right]^2\left(\frac{\partial V}{\partial P}\right)_T + kC_V T^2$$

$$\left\langle \Delta N^2 \right\rangle = -\frac{kTN^2}{V^2}\left(\frac{\partial V}{\partial P}\right)_{T,N} = kT\left(\frac{\partial N}{\partial m}\right)_{T,V}$$

Other quantities are found using basic quantities by expanding into differential of first/second order and then squaring it and putting into the probability distribution or by using the assumption that the fluctuation is small enough to use $\Delta f \cong \frac{\partial f}{\partial x}\Delta x$ while Δx is known

Mean values $\begin{bmatrix} \langle x^2 \rangle & \langle xy \rangle \\ \langle yx \rangle & \langle y^2 \rangle \end{bmatrix} = \frac{1}{ac - b^2}\begin{bmatrix} c & -b \\ -b & a \end{bmatrix}$

while $w \propto \exp\left[-\frac{1}{2}\left(ax^2 + 2bxy + cy^2\right)\right]$

Correlation is calculated using the cross terms of the matrice

Density correlation

$$\left\langle \Delta n_1 \Delta n_2 \right\rangle = \bar{n}\left[n(r) + d(r_2 - r_1)\right]$$

$$\int n(r)\,dV = \frac{\left\langle \Delta N^2 \right\rangle}{N} - 1 = -\frac{kTN}{V^2}\left(\frac{\partial V}{\partial P}\right)_T - 1$$

$$\int n(r)\,e^{-ikr}dr = V\frac{\left\langle n_k^2 \right\rangle}{\bar{n}} - 1$$

Degenerate gas

Fermions $\left\langle \Delta n_k^2 \right\rangle = \bar{n}_k\,(1 - \bar{n}_k)$

Bosons $\left\langle \Delta n_k^2 \right\rangle = \bar{n}_k\,(1 + \bar{n}_k)$

while n_k density of quantum states

Time Dependent Fluctuations

Quasi-Stationary state is a state in witch th macroscopic quantities don't change with tim and are large comparable to fluctuations.

Stationary State all fluctuations have reache their constant value around the mean value.

Langevin theory $M\frac{dx}{dt} = -\frac{x}{B} + F(t)$ $\bar{F}(t) = 0$

or $\frac{dx}{dt} = -\frac{x}{t} + A(t)$ $t = BM$

Mobility $D = BkT$

Random force correlation $K_F(s) = \left\langle F(t)F(t+s) \right\rangle = \frac{2}{t}\left\langle x^2 \right\rangle d(t$

for $x \to v(t)$ $K_F(s) = \frac{6kT}{B}d(s)$

$$K_A(s) = \frac{6kT}{BM^2}d(s)$$

Quantities Correlation

$$K_x(s) = \left\langle x^2 \right\rangle e^{-|s|/t}$$

when 'x' in stationary state

for $x \to v(t)$ $K_v(s) = \frac{3kT}{M}e^{-|s|/t}$

using equipartition value $\left\langle v^2 \right\rangle = \frac{3kT}{M}$

Fluctuations time dependence $t \gg t$

$$\left\langle x^2(t) \right\rangle = x^2(0)e^{-2t/t} + C\frac{t}{2}\left(1 - e^{-2t/t}\right)$$

for position $\frac{d^2}{dt^2}\left\langle r^2 \right\rangle + \frac{1}{t}\frac{d}{dt}\left\langle r^2 \right\rangle = 2\left\langle v^2 \right\rangle$

Dissipation-Fluctuation Theorem

$$D = \frac{1}{6}\int_{-\infty}^{\infty} K_V(s)\,ds \qquad \frac{1}{B} = \frac{M^2}{6kT}\int_{-\infty}^{\infty} K_A(s)\,sa$$

Spectral Analysis $x(t) = \frac{1}{2p}\int_{-\infty}^{\infty} S(w)\,e^{-iwt}dw$

Furrier component $x_w = \int_{-\infty}^{\infty} x(t)\,e^{iwt}dt$

Spectrum correlation $\left\langle x_{w_1} x_{w_2} \right\rangle = 2p\left(x^2\right)_w d(w_2 - w_1)$

Power spectrum $S(w) = \left(x^2\right)_w = \int_{-\infty}^{\infty} K_x(t)\,e^{iwt}dt$

or $K_x(t) = \frac{1}{2p}\int_{-\infty}^{\infty} S(w)\,e^{iwt}dw$

in particular $K(0) = \left\langle x^2 \right\rangle = \frac{1}{2p}\int_{-\infty}^{\infty} S(w)\,dw$

Quasi-Stationary Spectrum $S(w) = \frac{2t\left\langle x^2 \right\rangle}{t^2 w^2 + 1}$ $w \ll t^{-1}$

Nyquist Relation $\left\langle I^2 \right\rangle = \frac{4kT}{R}\Delta f$ for white noise

etc. $wt \ll 1$

Generalized Susceptibility

Response to 'force' $x(t) = \hat{a} f = \int_0^\infty a(t)\, f(t-t)\, dt$

Furrier components relation $x_w = a\, f_w$

Generalized Susceptibility $a(w) = a_1(w) + i \cdot a_2(w)$ $\quad a_1(-w) = a_1(w) \quad a_2(-w) = -a_2(w)$

Energy Dissipation per cycle $L = \frac{1}{2} a_2(w)\, w f_0^2$

Calculation

$$a_1 = \frac{2}{p} \int_0^\infty a_2(m) \frac{m \cdot dm}{m^2 - w^2}$$

$$a_2 = \frac{2}{p} \int_0^\infty a_1(m) \frac{w \cdot dm}{w^2 - m^2}$$

$a(w)$ is found by writing the appropriate relation between the force and the reaction to force.

Critical Indices

Order parameter (b) - found using the appropriate gibbs potential expansion and its stability factors + assuming linearity of the A coeffiecent.

Heat Capacity (a) - found using the expansion for the entropy $S = -\left(\frac{\partial G}{\partial T}\right)_\Psi \cong S_0 - \frac{\partial A}{\partial T}\Psi^2 + ..$ and the dependence on temperature of the order parameter.

Susceptibility (g) - found from writing exact formula $c^{-1} = \partial^2 G / \partial \Psi^2$ and putting the values of the order parameter at different temp. areas.

External Field (d) - like susc., but using $h = \partial G / \partial \Psi$

Summary of critical indices for different transitions

Indices	II	I
a	0	$a = \frac{1}{2} \quad a' = 0$
b	$\frac{1}{2}$	$\frac{1}{4}$
g	1	1
d	3	5

* a' means indice for $T > T_c$

Transitions with External Field

Second kind no transitions is observed in finite temperature. The order parameter tends to zero with the increasing of the temp.

First kind the transition is observed until finite critical external field.

Critical Field found from the condition of pitul point on the $h(\Psi)$ graph. means $\partial h / \partial \Psi = \partial^2 h / \partial \Psi^2 = 0$

generally $h_{cr} = 16 C_c \left(-\frac{B_{cr}}{5C_c}\right)^{\frac{5}{2}}$ and

$$A_{cr} = \frac{3B_c^2}{5C} \qquad T_c = T_0 + \frac{3B_c^2}{5aC_c}$$

Phase Transitions

Clausius-Clapeyron formula $\frac{dP}{dT} = \frac{Q}{T \Delta V}$

Ehrenfest formula $\frac{dP}{dT} = \frac{C_{P1} - C_{P2}}{TV(a_1 - a_2)} \quad a = \frac{1}{V}\left(\frac{\partial V}{\partial T}\right)_P$

Tinza theory $dG = \frac{1}{2}\sum_{i,j} (\partial_{ij} U)\, dx_i dx_j$

when $P_i = \frac{\partial U}{\partial x_i} \qquad \partial_{ij} U = \left(\frac{\partial P_i}{\partial x_j}\right)_{X_k}$

Stability $dG > 0 \;\rightarrow\; \det(\partial_{ij} U) > 0$

Phase Trans. $dG = 0 \;\rightarrow\; \det(\partial_{ij} U) = 0$

$$V_{ij} = \left(\frac{\partial x_i}{\partial P_j}\right)_{P_k} = (\partial_{ij} U)^{-1} \qquad V_{ij} \rightarrow \infty$$

Landau theory $G = G_0 + A(T,P)\Psi^2 + B(P)\Psi^4 + ...$

Order parameter $\Psi = 0 \quad T > T_c$

$\Psi \neq 0 \quad T < T_c$

External field $h = \frac{\partial G}{\partial \Psi} \qquad dU = h \cdot d\Psi$

Susceptibility $c = \frac{\partial \Psi}{\partial h} \qquad c^{-1} = \frac{\partial^2 G}{\partial \Psi^2}$

Second Order Transition

Stability assumptions **1)** $h = 0$ **2)** $c^{-1} > 0$

Coefficients in trans. $A = 0 \quad B > 0$

Linearity Assumption $A \cong a(T - T_c)$

Order Parameter $\Psi^2 = -\frac{A}{2B} \quad T < T_c$

First Order Transition

Gibbs potential $G = \Psi^2 (A + B\Psi^2 + C\Psi^4)$

Stability assumptions **1)** $G = 0$ **2)** $h = 0$

Coefficients in trans. $A_c > 0 \quad B_c < 0$

$$B_c^2 = 4 A_c C_c \qquad \Psi_c^2 = -\frac{2A}{B}$$

Order Parameter $\Psi^2 = \frac{-B \pm \sqrt{B^2 - 3CA}}{3C} \quad T < T_c$

T_S **Temperature** $B_S^2 = 3 A_S C_S$

only one solution for the func. $\Psi(T)$

T_0 **Temperature** $\Psi_0 = 0 \quad \Psi_0^2 = -2B_0$

Linearity Assum. $A \cong \begin{cases} a(T - T_0) & T < T_c \\ a(T_S - T) & T > T_c \end{cases}$

Critical Temp. $T_c = \begin{cases} T_0 + \frac{B_0^2}{3C_0 a} \\ T_s - \frac{B_S^2}{3C_S a} \end{cases}$

TCP (Tricritical Point) $A = 0 \quad B = 0$