Magnetostatics in Ponderable Media

Magnetization $\vec{M} = \sum_i N_i \langle \vec{m}_i \rangle$

Magnetic permeability m

Magnetic susceptibility $c_M = \partial M / \partial H$

Magnetic Field $\vec{H} = \frac{1}{m_0}\vec{B} - \vec{M}$

Maxwell equations in media $\vec{\nabla} \times \vec{H} = \vec{J}$ $\quad \vec{\nabla} \cdot \vec{B} = 0$

isotropic & linear media $\vec{B} = m\vec{H}$ $\quad \vec{M} = c_M \vec{H}$

Boundary conditions $\left(\vec{B}_2 - \vec{B}_1\right) \cdot \vec{n}_{21} = 0$

$\vec{n}_{21} \times \left(\vec{H}_2 - \vec{H}_1\right) = \vec{a}$

or $\vec{B}_2 \cdot \vec{n}_{21} = \vec{B}_1 \cdot \vec{n}_{21}$ $\quad \vec{B}_2 \times \vec{n}_{21} = \frac{m_2}{m_1} \vec{B}_1 \times \vec{n}_{21}$

$\vec{H}_2 \cdot \vec{n}_{21} = \frac{m_2}{m_1} \vec{H}_1 \cdot \vec{n}_{21}$ $\quad \vec{H}_2 \times \vec{n}_{21} = \vec{H}_1 \times \vec{n}_{21}$

Magnetic Scalar Potential $J = 0 \;\rightarrow\; \vec{H} = -\nabla \Phi_M$

linear media $\nabla^2 \Phi_M = -r_M$

Effective magnetic charge density $r_M = -\vec{\nabla} \cdot \vec{M}$

Solution for Scalar potential

$$\Phi_M\left(\vec{r}\right) = -\frac{1}{4p}\int_V \frac{\vec{\nabla}' \cdot \vec{M}\left(\vec{r}'\right)}{\left|\vec{r} - \vec{r}'\right|} d^3 r' + \frac{1}{4p}\oint_S \frac{n' \cdot \vec{M}\left(\vec{r}'\right)}{\left|\vec{r} - \vec{r}'\right|} da'$$

Energy of the field $W = \frac{1}{2}\int \vec{H} \cdot \vec{B}\; d^3 r = \frac{1}{2}\int \vec{J} \cdot \vec{A}\; d^3 r$

Change in Energy $W = \frac{1}{2}\int \vec{M} \cdot \vec{B}_0\, d^3 r$

putting object with $M \neq 0$ into magnetic field

Useful Relations

Spherical $\leftrightarrow$ Cartesian

$\hat{r} = \cos j \sin q\; \hat{x} + \sin j \sin q\; \hat{y} + \cos q\; \hat{z}$

$\hat{q} = \cos j \cos q\; \hat{x} + \sin j \cos q\; \hat{y} - \sin q\; \hat{z}$

$\hat{j} = -\sin j\; \hat{x} + \cos j\; \hat{y}$

$\hat{x} = \cos j \sin q\; \hat{r} + \cos j \cos q\; \hat{q} - \sin j\; \hat{j}$

$\hat{y} = \sin j \sin q\; \hat{r} + \sin j \cos q\; \hat{q} + \cos j\; \hat{j}$

$\hat{z} = \cos q\; \hat{r} - \sin q\; \hat{q}$

Polar $\leftrightarrow$ Cartesian

$\hat{r} = \cos q\; \hat{x} + \sin q\; \hat{y}$ $\quad \hat{q} = -\sin q\; \hat{x} + \cos q\; \hat{y}$

$\hat{x} = \cos q\; \hat{r} - \sin q\; \hat{q}$ $\quad \hat{y} = \sin q\; \hat{r} + \cos q\; \hat{q}$

Moving body current $\vec{J} = r \cdot \vec{v}$

Rotating body current $\vec{J} = r \cdot \left(\vec{w} \times \vec{r}\right)$

Rotating hollow sphere $J_j = wRs \sin q \cdot d\left(r - R\right)$

Rotating hollow cylinder $J_j = wRs \cdot d\left(r - R\right)$

Infinite wire $B_j\left(r\right) = \frac{m_0 I}{2pr}$

Force between 2 wires $\frac{dF}{dl} = \frac{m_0}{2p}\frac{I_1 I_2}{d}$

Finite Solenoid $B_z = \frac{1}{2} m_0 NI\left(\cos q_1 + \cos q_2\right)$

Time dependent fields

Magnetic flux $F = \int_S \vec{B} \cdot \vec{n}\; da$

Electromotive force $e = \oint_C \vec{E} \cdot d\vec{l}$

Faraday induction law $e = -dF/dt$

$$\vec{\nabla} \times \vec{E} + \frac{\partial \vec{B}}{\partial t} = 0$$

Coefficients of Self- & Mutual Inductances

$$W = \frac{1}{2}\sum_{i=1}^{N} L_i I_i^2 + \sum_{i=1}^{N}\sum_{j>i}^{N} M_{ij} I_i I_j$$

Mutual inductance $M_{ij} = F_{ij} / I_j = M_{ji}$

F_{ij} is flux from "j" linked with circuit "i"

EMF by inductance coefficients $e_{ij} = -dF_{ij}/dt$

Constant geometry $e_{ij} = -M_{ij}\frac{dI_i}{dt}$ $\quad e_{ii} = -L_i \frac{dI_i}{dt}$

Constant currents $e_{ij} = -I_j \frac{dM_{ij}}{dt}$ $\quad e_{ii} = -I_i \frac{dL_i}{dt}$

Maxwell Eqt. and Gauge Transformations

Maxwell Eqt.

$\vec{\nabla} \cdot \vec{D} = r$ $\quad \vec{\nabla} \cdot \vec{B} = 0$

$\vec{\nabla} \times \vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t}$ $\quad \vec{\nabla} \times \vec{E} + \frac{\partial \vec{B}}{\partial t} = 0$

in vacuum, no sources

$\vec{\nabla} \cdot \vec{B} = 0$ $\quad \vec{\nabla} \times \vec{B} = e_0 m_0 \frac{\partial \vec{E}}{\partial t}$

$\vec{\nabla} \cdot \vec{E} = 0$ $\quad \vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$

Wave Equations $\nabla^2 \vec{B} = m_0 e_0 \frac{\partial^2 \vec{B}}{\partial t^2}$ $\quad \nabla^2 \vec{E} = m_0 e_0 \frac{\partial^2 \vec{E}}{\partial t^2}$

Velocity of light $c = 1/\sqrt{m_0 e_0}$

Gauge Transformations $\vec{A}' = \vec{A} + \nabla \Lambda$ $\quad \Phi' = \Phi - \frac{\partial \Lambda}{\partial t}$

Delambertian $\Box^2 \equiv \nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}$

Lorenz gauge $\vec{\nabla} \cdot \vec{A} + \frac{1}{c^2}\frac{\partial \Phi}{\partial t} = 0$

giving $\Box^2 \Phi = -r/e_0$ $\quad \Box^2 \vec{A} = -m_0 \vec{J}$

Coulomb gauge $\vec{\nabla} \cdot \vec{A} = 0$

giving $\nabla^2 \Phi = -r/e_0$ $\quad \Box^2 \vec{A} = -m_0 \vec{J}_t$

Poynting Vector $\vec{S} = \vec{E} \times \vec{H}$

Magnetostatics

Continuity equation $\frac{\partial r}{\partial t}+\vec{\nabla}\cdot\vec{J}=0$

in magnetostatics $\vec{\nabla}\cdot\vec{J}=0$

Biot-Savart Law $d\vec{B}=\frac{m_0 I}{4p}\frac{\left(d\vec{l}\times\vec{r}\right)}{r^3}$

Moving Charge Field $\vec{B}=\frac{m_0 q}{4p}\frac{\vec{v}\times\vec{r}}{r^3}$

First Ampere's Law $d\vec{F}=I_1\left(d\vec{l}_1\times\vec{B}\right)$

Force $\vec{F}=\int\vec{J}\times\vec{B}\ d^3r$

Torque $\vec{N}=\int\vec{r}\times\left(\vec{J}\times\vec{B}\right)\ d^3r$

Magnetic induction $\vec{B}(\vec{r})=\frac{m_0}{4p}\int\vec{J}(\vec{r}\,')\times\frac{\vec{r}-\vec{r}\,'}{\left|\vec{r}-\vec{r}\,'\right|^3}d^3r'$

$\vec{\nabla}\cdot\vec{B}=0 \qquad \vec{\nabla}\times\vec{B}=m_0\vec{J}$

Second Ampere's Law $\oint_C\vec{B}\cdot d\vec{l}=m_0 I$

Vector Potential $\vec{A}(\vec{r})=\frac{m_0}{4p}\int\frac{\vec{J}(\vec{r}\,')}{\left|\vec{r}-\vec{r}\,'\right|}d^3r'+\nabla\Psi(\vec{r})$

$\vec{B}=\vec{\nabla}\times\vec{A} \qquad \nabla\left(\vec{\nabla}\cdot\vec{A}\right)-\nabla^2\vec{A}=m_0\vec{J}$

Magnetic moment $\vec{m}=\frac{1}{2}\int\vec{r}\,'\times\vec{J}(\vec{r}\,')\ d^3r'$

current loop $\vec{m}=\frac{I}{2}\oint\vec{r}\times d\vec{l}$

planar loop $m=I\cdot(Area)$

number of particles $\vec{m}=\frac{q}{2M}\vec{L}$

where $\vec{L}$ total angular momentum

Dipole Potential $\vec{A}(\vec{r})=\frac{m_0}{4p}\frac{\vec{m}\times\vec{r}}{r^3}$

Dipole induction $\vec{B}(\vec{r})=\frac{m_0}{4p}\left[\frac{3\vec{n}(\vec{n}\cdot\vec{m})-\vec{m}}{r^3}\right]$

Force on dipole $\vec{F}=\nabla\left(\vec{m}\cdot\vec{B}\right)$

Torque on dipole $\vec{N}=\int\left[\left(\vec{r}\,'\cdot\vec{B}\right)\vec{J}-\left(\vec{r}\,'\cdot\vec{J}\right)\vec{B}\right]d^3r'$

Potential energy of dipole $U=-\vec{m}\cdot\vec{B}$

Expanding Potential to Multipoles

Point charge expansion

$$\frac{1}{\left|\vec{r}-\vec{r}\,'\right|}=4p\sum_{l,m}\frac{1}{2l+1}\frac{r_<^l}{r_>^{l+1}}Y_{lm}^*(q',f')Y_{lm}(q,f)$$

Potential $\Phi=\frac{1}{e_0}\sum_{l,m}\frac{1}{2l+1}q_{lm}\frac{Y_{lm}(q,f)}{r^{l+1}}$

or $\Phi=\frac{1}{4pe_0}\left[\frac{q}{r}+\frac{\vec{p}\cdot\vec{r}}{r^3}+\frac{1}{2}\sum_{i,j}Q_{ij}\frac{x_i x_j}{r^5}+\cdots\right]$

Multipoles $q_{lm}=\int Y_{lm}^*(q,f)r^l\,r(\vec{r})\ d^3r$

Dipole $\vec{p}=\int\vec{r}\,r(\vec{r})\ d^3r$

Dipole Field $\vec{E}=\frac{3\vec{n}(\vec{p}\cdot\vec{n})-\vec{p}}{4pe_0\left|\vec{r}-\vec{r}_0\right|^3}$

Quadrupole $Q_{ij}=\int\left(3x_i x_j-r^2 d_{ij}\right)r(\vec{x})\ d^3x$

* Multipole with order "n" is independent on the origin of coordinates only if all the multipoles with lower order vanish.

Electrostatics in Ponderable Media

Dipole moment density $\vec{P}=N\left\langle\vec{p}_{mol}\right\rangle$

Electric Displacement $\vec{D}=e_0\vec{E}+\vec{P}$

Electric susceptibility $c_e=\partial P/\partial E$

Dielectric constant $e/e_0=1+c_e$

Electric permittivity $e=e_0(1+c_e)$

Maxwell equations in media $\vec{\nabla}\times\vec{E}=0 \quad \vec{\nabla}\cdot\vec{D}=r$

isotropic & linear media $\vec{D}=e\vec{E} \quad \vec{P}=e_0 c_e\vec{E}$

+ homogeneous media $\vec{\nabla}\cdot\vec{E}=\frac{r}{e}$

Boundary conditions $\left(\vec{D}_2-\vec{D}_1\right)\cdot\vec{n}_{21}=s \quad E_{\|1}=E_{\|2}$

Induced charge density $s_{ind}=-\left(\vec{P}_2-\vec{P}_1\right)\cdot\vec{n}_{21}$

Internal Field $\vec{E}_i=\frac{\vec{P}}{3e_0}+\vec{E}_{near}$

Average polarizability $\left\langle\vec{p}_{mol}\right\rangle=e_0 g_{mol}\left(\vec{E}+\vec{E}_i\right)$

Induced $\left\langle p_{mol}\right\rangle_{ind}=\frac{e^2}{mw_0^2}E$

Permanent moment $\left\langle p_{mol}\right\rangle_0\cong\frac{1}{3}\frac{p_0^2}{k_B T}E$

Molecular polarizability $g_{mol}=\frac{3}{N}\frac{e/e_0-1}{e/e_0+2}$

Classical model $g_{mol}\cong g_{ind}+\frac{1}{3e_0}\frac{p_0^2}{k_B T}$

Dielectrics Energy $W=\frac{1}{2}\int\vec{E}\cdot\vec{D}\ d^3r$

Change in Energy $W=-\frac{1}{2}\int\vec{P}\cdot\vec{E}_0 d^3r$

putting object with $M\neq 0$ into magnetic field

Electrostatics

Electric field $\vec{E}(\vec{r}) = \frac{1}{4pe_0}\int r(\vec{r}')\frac{\vec{r}-\vec{r}'}{|\vec{r}-\vec{r}'|^3}d^3r'$

Gauss Law $\int_S \vec{E}\cdot\vec{n}da = \frac{Q}{e_0} = \frac{1}{e_0}\int r(\vec{r})dV$

$\vec{\nabla}\cdot\vec{E} = \frac{r}{e_0}$

Electric potential $\Phi(\vec{r}) = \frac{1}{4pe_0}\int\frac{r(\vec{r}')}{|\vec{r}-\vec{r}'|}d^3r' + const$

$E = -\nabla\Phi \qquad \vec{\nabla}\times\vec{E} = 0$

Potential Energy

$$W = -\int_A^B \vec{F}\cdot d\vec{l} = q[\Phi(B)-\Phi(A)]$$

Electrostatic Potential Energy

$$W = \frac{1}{2}\int r(\vec{r})\Phi(\vec{r})\ d^3r = \frac{e_0}{2}\int|\vec{E}|^2\ d^3r$$

Energy density $\qquad w = \frac{e_0}{2}|\vec{E}|^2$

Electric force $\qquad \vec{F} = -\frac{\partial W}{\partial\vec{r}}$

Force per unit area in conductor $\qquad f = \frac{s^2}{2e_0}$

Boundary conditions $\qquad E_{||1} = E_{||2} \quad E_{\perp 2} - E_{\perp 1} = \frac{s}{e_0}$

Dipole layer potential $\qquad \Phi(\vec{r}) = -\frac{1}{4pe_0}\int_S D(\vec{r}')d\Omega'$

where $D(\vec{r}) = \lim_{d\to 0} s(\vec{r})\cdot d(\vec{r})$ & Ω is the solid angle from the observation point

Boundary: $\qquad \Phi_2 - \Phi_1 = \frac{D}{e_0}$

Poisson & Laplace Eqt. $\qquad \nabla^2\Phi = -r/e_0 \qquad \nabla^2\Phi = 0$

Point source $\quad \nabla^2\left(\frac{1}{|\vec{r}-\vec{r}'|}\right) = -4pd(\vec{r}-\vec{r}')$

Capacitance $\qquad C \equiv \frac{Q}{\Delta V}$

Orthogonal Function Proprieties

Legendre functions

$$P_l(x) = \frac{1}{2^l l!}\frac{d^l}{dx^l}(x^2-1)^l$$

$$\int_{-1}^{1} P_{l'}(x)P_l(x)dx = \frac{2}{2l+1}d_{l'l} \qquad \int P_l dx = \frac{P_{l+1}-P_{l-1}}{2l+1}$$

$$P_l(1) = 1 \qquad P_l(-1) = (-1)^l$$

$$P_l(0) = (-1)^{l/2}\frac{1\cdot 3\cdot 5\cdots(l-1)}{2\cdot 4\cdot 6\cdots n} \quad \text{n-even}$$

Spherical harmonics

$$Y_{lm}(p-q, f+p) = (-1)^l Y_{lm}(q,f)$$

$$Y_{lm}(p-q, f) = (-1)^{l+m} Y_{lm}(q,f)$$

$$Y_{l,-m} = (-1)^m Y_{lm}^*$$

$$Y_{lm}(q, f+p) = (-1)^m Y_{lm}(q,f)$$

Green function

Green first identity

$$\int_V\left(f\nabla^2 y + \nabla f\cdot\nabla y\right)d^3x = \oint_S f\frac{\partial y}{\partial n}da$$

Green second identity (Green's Theorem)

$$\int_V\left(f\nabla^2 y - y\nabla^2 f\right)d^3x = \oint_S\left[f\frac{\partial y}{\partial n} - y\frac{\partial f}{\partial n}\right]da$$

Green Potential

$$\Phi(\vec{r}) = \frac{1}{4pe_0}\int_V\frac{r(\vec{r}')}{|\vec{r}-\vec{r}'|}d^3r' + \frac{1}{4p}\oint_S\left[\frac{1}{|\vec{r}-\vec{r}'|}\frac{\partial\Phi}{\partial n'} - \Phi\frac{\partial}{\partial n'}\left(\frac{1}{|\vec{r}-\vec{r}'|}\right)\right]$$

Green Function

$$G(\vec{r},\vec{r}') = \frac{1}{|\vec{r}-\vec{r}'|} + F(\vec{r},\vec{r}') \qquad \nabla'^2 F = 0$$

Dirichlet boundary $\quad G_D = 0$ on S

$$\Phi(\vec{r}) = \frac{1}{4pe_0}\int_V r(\vec{r}')G_D(\vec{r},\vec{r}')\ d^3r' - \frac{1}{4p}\oint_S\Phi(\vec{r}')\frac{\partial G_D}{\partial n'}da'$$

Neumann boundary $\quad \frac{\partial G_N}{\partial n'} = -\frac{4p}{S}$ $\quad$ on S

$$\Phi(\vec{r}) = \langle\Phi\rangle_S + \frac{1}{4pe_0}\int_V r(\vec{r}')G_N(\vec{r},\vec{r}')\ d^3r' + \frac{1}{4p}\oint_S G_N\frac{\partial\Phi}{\partial n'}da'$$

Green's reciprocation theorem

$$\int_V r\Phi' d^3r + \int_S s\Phi'\ da = \int_V r'\Phi d^3r + \int_S s'\Phi\ da$$

Orthogonal Functions Expansion

Rectangular $\qquad \Phi = e^{\pm ia x}e^{\pm ib y}e^{\pm z\sqrt{a^2+b^2}}$

Polar $\quad \Phi(r,f) = a_0 + b_0\ln r + \sum_{n=1}^{\infty}\left(a_n r^n + b_n r^{-n}\right)(A\sin nf + B\cos nf)$

Spherical Symmetrical Azimuthally

$$\Phi(r,q) = \sum_{l=0}^{\infty}\left[A_l r^l + B_l r^{-(l+1)}\right]P_l(\cos q)$$

General Spherical

$$\Phi(r,q,f) = \sum_{l=0}^{\infty}\sum_{m=-l}^{l}\left[A_{lm}r^l + B_{lm}r^{-(l+1)}\right]Y_{lm}(q,f)$$

Point source $\quad \frac{1}{|\vec{r}-\vec{r}'|} = 4p\sum_{l=0}^{\infty}\sum_{m=-l}^{l}\frac{1}{2l+1}\frac{r_<^l}{r_>^{l+1}}Y_{lm}^*(q',f')Y_{lm}(q,f)$

Low orders of Orthogonal functions

$$Y_{00} = \frac{1}{\sqrt{4p}} \qquad Y_{20} = \sqrt{\frac{5}{4p}}\left(\frac{3}{2}\cos^2 q - \frac{1}{2}\right)$$

$$Y_{11} = -\sqrt{\frac{3}{8p}}\sin q\, e^{if} \qquad Y_{21} = -\sqrt{\frac{15}{8p}}\sin q\cos q\, e^{if}$$

$$Y_{10} = \sqrt{\frac{3}{4p}}\cos q \qquad Y_{22} = \frac{1}{4}\sqrt{\frac{15}{2p}}\sin^2 q\, e^{2if}$$

$$P_0 = 1 \qquad P_1 = x$$

$$P_2 = \frac{1}{2}(3x^2-1) \qquad P_3 = \frac{1}{2}(5x^3-3x)$$