# Electromagnetic Waves

**Maxwell Relations** $\vec{\nabla}\times\vec{E} = -\frac{\partial \vec{B}}{\partial t}$ $\vec{\nabla}\times\vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t}$

$\vec{\nabla}\cdot\vec{D} = 0$ $\vec{\nabla}\cdot\vec{B} = 0$

**Electric Displacement** $\vec{D} = \varepsilon_0\vec{E} + \vec{P} \overset{*}{=} \varepsilon_r\varepsilon_0\vec{E}$

* true only in linear and uniform media

**Magnetic Inductance** $\vec{B} = \mu_0\left(\vec{H} + \vec{M}\right) \overset{*}{=} \mu_r\mu_0\vec{H}$

* true only in linear and uniform media

**Velocity of Light** $v = \left(\varepsilon\mu\right)^{-\frac{1}{2}}$ $c = \left(\varepsilon_0\mu_0\right)^{-\frac{1}{2}}$

**Wave equations** $\nabla^2\vec{E} = \mu\varepsilon\frac{\partial^2\vec{E}}{\partial t^2}$ $\nabla^2\vec{H} = \mu\varepsilon\frac{\partial^2\vec{H}}{\partial t^2}$

**Wave number/frequency** $k = \frac{2\pi}{\lambda}$ $\omega = kc = 2\pi f$

**Index of Refraction** $n = \frac{c}{v} = \sqrt{\varepsilon}$

**Plane Wave** $\vec{E} = \vec{E}_0 \exp\left[i\left(\omega t - \vec{k}\cdot\vec{r}\right)\right]$

**Radial Wave** $\vec{E} = \vec{E}_0\frac{A}{r}\exp\left[i\left(\omega t - kr\right)\right]$

**Medium Impedance** $\frac{E}{H} = \sqrt{\frac{\mu}{\varepsilon}} = Z$ $Z_0 = 377\Omega$

**Field relations** $\vec{k}\perp\vec{E}$ $\vec{k}\perp\vec{H}$ $\vec{H}\perp\vec{E}$

**Poynting vector** $\vec{S} = \vec{E}\times\vec{H}$ power per unit area

**Poynting theorem** flow of energy via closed surface

$$\oint_S\left(\vec{E}\times\vec{H}\right)\cdot d\vec{a} = \int_V \vec{E}\cdot\vec{J} + \frac{\partial}{\partial t}\left(\frac{\varepsilon}{2}E^2 + \frac{\mu}{2}H^2\right) + \vec{E}\frac{\partial\vec{P}}{\partial t} + \mu\vec{H}\frac{\partial\vec{M}}{\partial t}dV$$

**Energy Consumption** $\frac{1}{V}\left\langle P\right\rangle = \frac{1}{2}\omega\varepsilon_0\left|E\right|^2\operatorname{Im}\left(\chi_e\right)$

**Wave Intensity** $I = \left\langle\vec{S}\right\rangle$ if $T \gg \frac{2\pi}{\omega}$

while “T” is the interval between measurements

**Boundary conditions** $E_{1\parallel} = E_{2\parallel}$ $\varepsilon_1E_{1\perp} = \varepsilon_2E_{2\perp}$

$H_{1\parallel} = H_{2\parallel}$ $\mu_1H_{1\perp} = \mu_2H_{2\perp}$

**Snell Law** $n_i\sin\theta_i = n_t\sin\theta_t$

**Transmittance/Reflection**

$$r_{TE} = \frac{n_i\cos\theta_i - n_t\cos\theta_t}{n_i\cos\theta_i + n_t\cos\theta_t} \quad t_{TE} = \frac{2n_i\cos\theta_i}{n_i\cos\theta_i + n_t\cos\theta_t}$$

$$r_{TM} = \frac{n_t\cos\theta_i - n_i\cos\theta_t}{n_i\cos\theta_t + n_t\cos\theta_i} \quad t_{TM} = \frac{2n_i\cos\theta_i}{n_i\cos\theta_t + n_t\cos\theta_i}$$

$$T = t^2\frac{n_t\cos\theta_t}{n_i\cos\theta_i} \quad R = r^2 \quad R + T = 1$$

**Brewster Angle** $\vartheta = \arctan\left(\frac{n_t}{n_i}\right) \overset{n_t\approx 1.5}{=} 56^\circ$

**Polarization** *Linear* $E_{0x} = E_{0y}$ $\Delta\varphi = 0$

*Circular* $E_{0x} = E_{0y}$ $\Delta\varphi = \frac{\pi}{2}$

*Elliptic* else

# Ray Optics

**Conventions** Distances from the right of the system are positive & from left are negative. Curvature radius with center from the right is positive. Angles beyond x-axis are positive.

**v** – distance from system to image

**u** – distance from system to object

**Complex Matrices** matrix of a system with $1,2...N$ components in a raw is $M = M_N\cdot\ldots M_2\cdot M_1$

**Determinant propriety** $\det M = \begin{vmatrix} A & B \\ C & D \end{vmatrix} = \frac{n_{in}}{n_{out}}$

**Transmition in media** $\begin{pmatrix} r_{out} \\ \hat{r}_{out} \end{pmatrix} = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}\begin{pmatrix} r_{in} \\ \hat{r}_{in} \end{pmatrix}$

**Planar Surface** $\begin{pmatrix} r_{out} \\ \hat{r}_{out} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & \frac{n_1}{n_2} \end{pmatrix}\begin{pmatrix} r_{in} \\ \hat{r}_{in} \end{pmatrix}$

**Spherical Surface** $\begin{pmatrix} r_{out} \\ \hat{r}_{out} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ \left(\frac{n_1}{n_2} - 1\right)\frac{1}{R} & \frac{n_1}{n_2} \end{pmatrix}\begin{pmatrix} r_{in} \\ \hat{r}_{in} \end{pmatrix}$

**Spherical Mirror** $\begin{pmatrix} r_{out} \\ \hat{r}_{out} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ \frac{2}{R} & 1 \end{pmatrix}\begin{pmatrix} r_{in} \\ \hat{r}_{in} \end{pmatrix}$

**Thin lens** $\begin{pmatrix} r_{out} \\ \hat{r}_{out} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -\frac{1}{f} & 1 \end{pmatrix}\begin{pmatrix} r_{in} \\ \hat{r}_{in} \end{pmatrix}$

*Focal point* $\frac{1}{f} = \frac{n_2 - n_1}{n_1}\left(\frac{1}{R_1} - \frac{1}{R_2}\right)$

**Paths of Rays**

1) ray through the center travels unchanged
2) parallel rays meet at the focal plane
3) parallel rays to the axis meet at the focus

**Imaging** if $\begin{pmatrix} r_{out} \\ \hat{r}_{out} \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix}\begin{pmatrix} r_{in} \\ \hat{r}_{in} \end{pmatrix}$ then $B = 0$

$$\begin{pmatrix} r_{out} \\ \hat{r}_{out} \end{pmatrix} = \begin{pmatrix} 1 & v \\ 0 & 1 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix}\begin{pmatrix} r_{in} \\ \hat{r}_{in} \end{pmatrix}$$

*Effective focus* $\tilde{f} = -\frac{1}{c}$

*Focal planes* $F_2 = -\frac{a}{c} = a\tilde{f}$ $F_1 = -\frac{d}{c} = d\tilde{f}$

*Principal planes* $u_p = \frac{1-d}{c} = \left(d-1\right)\tilde{f}$ $v_p = \frac{1-a}{c} = \left(a-1\right)\tilde{f}$

planes for witch the system acts like thin lens

*Imaging condition* $\frac{1}{u-u_p} + \frac{1}{v-v_p} = \frac{1}{\tilde{f}}$

**Newton equation** $\left(A - \frac{v}{\tilde{f}}\right)\left(D - \frac{u}{\tilde{f}}\right) = 1$

**Magnification** Linear $m = A = -\frac{v-v_p}{u-u_p}$

$m = 1$ for $v = v_p$ $u = u_p$

Angle $D = \frac{1}{m}$

**Image formation** $v > 0$ Real & inverted image

$v < 0$ Imaginary & straight image

$f < 0$ Inverted image

$f > 0$ Straight image

## Approximation Conditions

**Phase approximation** $kd = k\left(d^2 + x^2 + y^2\right)^{\frac{1}{2}} \approx kd\left(1 + \frac{1}{2}\frac{r^2}{d^2}\right) + \ldots$

d - *distance to screen*

$r$ - *distance off-axis of the aperture*

**Point source requirement** $D\rho / d_1 < \frac{1}{4}\lambda$

D - *effective diameter of the source*

$\rho$ - *radius of the aperture*

**Paraxial approximation** $\frac{x}{d}, \frac{y}{d} \ll 1 \qquad \sin x \approx x$

**Maximum divergence angle** $\alpha_{\max} = \arctan\left(\frac{a}{d}\right) \cong \frac{a}{d}$

"a" – *radius of the diffraction pattern*

**Fresnel number** $N_F \equiv \frac{a^2}{\lambda d}$

**Inverse Fresnel number** $N_F' \equiv \frac{b^2}{\lambda d}$

"b" – *radius of the aperture*

**Fresnel diffraction condition** $\frac{1}{4} N_F \alpha_{\max}^2 \ll 1$

**Fraunhofer diffraction conditions** $N_F' \ll 1 \quad N_F \ll 1$

## Interference

**Two monochromatic waves** $I = I_1 + I_2 + 2\sqrt{I_1 I_2}\cos\varphi$

*Phase difference* $\varphi = \mathbf{k}_1 \cdot \mathbf{r}_1 - \mathbf{k}_2 \cdot \mathbf{r}_2 + \left(\varepsilon_1 - \varepsilon_2\right)$

*Equal Amplitude* $I = 4I_0 \cos^2 \frac{\varphi}{2}$

**Beating** $I = I_1 + I_2 + 2\sqrt{I_1 I_2}\cos\left[2\pi\left(\nu_2 - \nu_1\right)t + \varepsilon\left(x, y\right)\right]$

*Fringe velocity* $\frac{dx}{dt} = -\frac{\left(d\varphi/dt\right)_x}{\left(d\varphi/dx\right)_t}$

## Furrier Transform Proprieties

**General proprieties**

$$FT\left[f\left(x - x_0\right)\right] = e^{-2\pi i \nu x_0} F\left(\nu\right)$$

$$FT\left[f\left(\frac{x}{x_0}\right)\right] = \left|x_0\right| F\left(x_0 \nu\right)$$

$$FT\left[f\left(x\right) f\left(y\right)\right] = F\left(\nu_x\right) F\left(\nu_y\right)$$

$$FT\left[f_1 \otimes f_2\right] = F_1\left(\nu\right) \cdot F_2\left(\nu\right)$$

$$FT\left[FT\left[f\left(x, y\right)\right]\right] = f\left(-x, -y\right)$$

$f\left(x\right)$ symmetrical $\rightarrow F\left(-\nu\right) = F^*\left(\nu\right)$

$f\left(x\right)$ real & symmetrical $\rightarrow F\left(\nu\right)$ also

$\boxed{F\left(\bullet \quad \bullet\right) = F\left(\bullet\right) \cdot F\left(\cdot \quad \cdot\right)}$

**Useful transforms**

$FT\left[rect\left(\frac{x}{a}\right)\right] = a\,\mathrm{sinc}\left(a\nu_x\right) \equiv a\frac{\sin\left(\pi a \nu_x\right)}{\pi a \nu_x}$

while $rect\left(\frac{x}{a}\right) = 1 \quad \left|x\right| \le a/2$

$FT\left[circ\left(r\right)\right] = \frac{J_1\left(2\pi\nu_p\right)}{\nu_p}$ where $\nu_p^2 = \nu_x^2 + \nu_y^2$

**Useful Fresnel Integrals on axis**

*Circular hole* $I = \left|g\right|^2 = \frac{8A^2\pi^2}{k^2}\left(1 - \cos\frac{k\rho^2}{2z_0}\right)$

*Circular disc* $I = \frac{4A^2\pi^2}{k^2}$

## Furrier Optics

**Spatial Frequency** $\nu_x = \frac{k_x}{2\pi} = \frac{\mathrm{x}}{\lambda \mathrm{d}} \quad \nu_y = \frac{k_y}{2\pi} = \frac{\mathrm{y}}{\lambda \mathrm{d}}$

**Incident angles** $\sin\theta_x = \lambda\nu_x \quad \sin\theta_y = \lambda\nu_y$

in paraxial approximation $\theta_x = \lambda\nu_x \quad \theta_y = \lambda\nu_y$

**Spatial periods** $\Lambda_x = \nu_x^{-1} \quad \Lambda_y = \nu_y^{-1} \quad \Lambda_z = \nu_z^{-1}$

**Distraction by obstacle/Lens** $\theta = \frac{\lambda}{d}$

**Phase mask distraction** if $f\left(x, y\right) = e^{-2\pi i \phi\left(x, y\right)}$

then $\nu_x\left(x\right) = \frac{\partial\phi}{\partial x} \quad \nu_y\left(y\right) = \frac{\partial\phi}{\partial y}$

**Input-Output Relations** in linear shift-invariant system (<u>without magnification</u>)

$$G\left(\nu_x, \nu_y\right) = H\left(\nu_x, \nu_y\right) F\left(\nu_x, \nu_y\right)$$

$$g\left(x, y\right) = f\left(x, y\right) \otimes h\left(x, y\right)$$

**Transfer Function**

***Free space*** $H = \exp\left[-2\pi i\left(\frac{1}{\lambda^2} - \nu_x^2 - \nu_y^2\right)^{\frac{1}{2}} d\right]$

*Far field* $\nu_p^2 \le \lambda^{-2} \quad \rightarrow \quad \left|H\right| = 1$

where $\nu_p^2 = \nu_x^2 + \nu_y^2$

*Near field* $\nu_p^2 \ge \lambda^{-2} \quad \left|H\right| = e^{-2\pi d\sqrt{\frac{2}{\lambda}\left(\nu_p - \lambda^{-1}\right)}}$

here $\nu_p$ is a ***Cut-Off*** frequency

***Fresnel Approx***. $H = H_0 \exp\left[i\pi\lambda d\left(\nu_x^2 + \nu_y^2\right)\right]$

where $H_0 = \exp\left(-ikd\right)$

**Impulse-Response Function** a response of the system to point source at the origin ($\delta$ func.).

***Inverse FT of the Transfer function.***

*Free Space* $h\left(x, y\right) = h_0 \exp\left[-ik\frac{x^2 + y^2}{2d}\right]$

in Fresnel approx. while $h_0 = \frac{i}{\lambda d} e^{-ikd}$

*Infinite Opening Lens*

$$h\left(x, y\right) = h_1 h_2 \exp\left[-i\frac{k}{2f}\left(x^2 + y^2\right)\right]\delta\left(-\frac{x}{\lambda d_2}, -\frac{y}{\lambda d_2}\right)$$

*Finite Opening Lens* $h\left(x, y\right) = h_1 h_2 \hat{P}\left(-\frac{x}{\lambda d_2}, -\frac{y}{\lambda d_2}\right)$

while $\hat{P}$ is FT of the aperture (pupil) function and the varying phase was neglected.

**Diffraction pattern width**

*Circular*: $\Delta = 2r_1 = \frac{1.22\lambda f}{D}$ D - mask diameter

*Rectangular*: $\Delta_x = \frac{2\lambda f}{b_x}$ $b_x$ - "x" mask width

**Furrier Transform by Lens**

$$g\left(x, y\right) = \frac{i}{\lambda f} e^{-ik\left(f + d\right)} e^{i\pi\lambda\left(d - f\right)\frac{x^2 + y^2}{\left(\lambda f\right)^2}} F\left(\frac{x}{\lambda f}, \frac{y}{\lambda f}\right)$$

plane wave is focused at $x_0 = \lambda f \nu_x \quad \mathrm{y}_0 = \lambda f \nu_y$

**Fraunhofer Diffraction**

$$g\left(x, y\right) = h_0 \exp\left[-i\frac{\pi}{\lambda d}\left(x^2 + y^2\right)\right] F\left(\frac{x}{\lambda d}, \frac{y}{\lambda d}\right)$$