Prepared by Boris Milner

[illegible]

λ - Demand rate

K - Fixed cost

c - cost per unit (variable cost)

h - Holding Cost rate

Q - Order Quantity

G(Q) - Average <u>annual</u> cost (as a function of Q)

T - Cycle time

g(T) - Average <u>annual</u> cost (as a function of T)

$$Q = \lambda \cdot T \longleftrightarrow T = \frac{Q}{\lambda}$$

Total Cost: $K + cQ + h \cdot \left(\frac{1}{2} QT\right)$

$$G(Q) = \frac{K + CQ + h \cdot \left(\frac{1}{2} QT\right)}{T} \longleftarrow$$

עלות כוללת במחזור

$$T = \frac{Q}{\lambda}$$

$$G(Q) = \frac{K + CQ + h \cdot \frac{1}{2} \cdot Q \cdot \frac{Q}{\lambda}}{\frac{Q}{\lambda}} = \frac{K\lambda}{Q} + C\lambda + \frac{hQ}{2}$$

אנו מעוניינים במינימום לכן נגזור ונשווה ל- 0.

$$G'(Q) = -\frac{K\lambda}{Q^2} + \frac{h}{2} = 0 \implies Q^* = \sqrt{\frac{2K\lambda}{h}} \qquad T = \frac{Q}{\lambda}$$

$$\implies T^* = \sqrt{\frac{2K}{\lambda h}}$$

משתני ההחלטה אינם תלויים ב- C

$$G(\underline{\underline{Q^*}}) = \sqrt{2K\lambda h}$$

τ - Lead time

R - Reorder Point: net inventory level when ordering

(Q,R) - Ordering Policy: order a fixed quantity Q every time the inventory is at or below R. If $\tau < T \Rightarrow R = \lambda\tau$

When $\tau > T$: $R = \lambda \cdot \tau_r$ $\quad (\tau_r = \tau - m \cdot T)$ $\quad m = \left\lfloor \frac{\tau}{T} \right\rfloor$

For finite Production Rate: $H = (P - \lambda) T_1 \Rightarrow H = (P - \lambda)\left(\frac{Q}{P}\right)$

$$\implies H = Q\left(1 - \frac{\lambda}{P}\right)$$

H - Maximum level of on-hand Inventory

Holding Cost: $h\cdot\left(\frac{1}{2}\cdot H\cdot T\right) = h\cdot\frac{1}{2}\cdot Q\left(1-\frac{\lambda}{P}\right)T$

Define $h' = h\left(1-\frac{\lambda}{P}\right) \Rightarrow$ Holding cost $= h'\cdot\frac{1}{2}QT$

$$Q^* = \sqrt{\frac{2K\lambda}{h'}} = \sqrt{\frac{2K\lambda}{h\left(1-\frac{\lambda}{P}\right)}}$$

When Backorders allowed

$\hat{P}$ - backorder cost rate (per time unit)

P - backorder cost per unit

B - Maximum backorder level (units)

$$\left(\begin{array}{l} B = T_1(P-\lambda) \\ T_1 = \frac{B}{P-\lambda}\end{array}\right) \quad \left(\begin{array}{l} H = T_2(P-\lambda) \\ T_2 = \frac{H}{P-\lambda}\end{array}\right) \quad \left(\begin{array}{l} H = T_3\cdot\lambda \\ T_3 = \frac{H}{\lambda}\end{array}\right)$$

$$\left(\begin{array}{l} B = T_4\cdot\lambda \\ T_4 = \frac{B}{\lambda}\end{array}\right) \qquad T_1 + T_2 = \frac{Q}{P}$$

$H = Q - B - (T_1+T_2)\lambda$: Inventory is what you produce minus what you use.

$= Q - B - \frac{Q}{P}\lambda = Q\left(1-\frac{\lambda}{P}\right) - B$

$H = -B + (T_1+T_2)(P-\lambda)$: Inventory is where you started plus what you put in

$= -B + \frac{Q}{P}(P-\lambda) = Q\left(1-\frac{\lambda}{P}\right) - B$

order cost

K per cycle

Inventory cost

$h\cdot\frac{1}{2}\cdot(T_2+T_3)\cdot H = h\cdot\frac{1}{2}\cdot\left(\frac{H}{P-\lambda}+\frac{H}{\lambda}\right)\cdot H =$

$= \frac{1}{2}\cdot h\left(\frac{H\cdot\lambda + H(P-\lambda)}{\lambda(P-\lambda)}\right)\cdot H = \frac{1}{2}\cdot h\cdot\frac{HP}{\lambda(P-\lambda)}\cdot H =$

$= \frac{1}{2}\cdot h\cdot\frac{H^2}{\left(1-\frac{\lambda}{P}\right)\lambda}$

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Shortage cost

$\frac{1}{2}\hat{p}\cdot\frac{B^2}{(1-\frac{\lambda}{P})\lambda}$ Time and quantity dependent

$p\cdot B$ Quantity dependent

$$G(Q)=\frac{K+\frac{1}{2}h\cdot\frac{H^2}{(1-\lambda/P)\lambda}+\frac{1}{2}\hat{p}\cdot\frac{B^2}{(1-\lambda/P)\lambda}+p\cdot B}{T}$$ עלות קבועה

$$Q^*=\sqrt{\frac{2K\lambda}{h(1-\lambda/P)}-\frac{(p\lambda)^2}{h(h+\hat{p})}}\cdot\sqrt{\frac{h+\hat{p}}{\hat{p}}}$$

$$B^*=\frac{(hQ^*-p\lambda)\cdot(1-\frac{\lambda}{P})}{h+\hat{p}}$$

Reorder Point: $R=\lambda\tau-B$

If $P=\infty \Rightarrow R=\lambda\tau$

$$\text{Fill rate}=\frac{Q-B}{Q}$$

Sensitivity

$$\frac{G(Q^A)}{G(Q^*)}=\frac{1}{2}\left[\frac{Q^*}{Q^A}+\frac{Q^A}{Q^*}\right];\quad \frac{g(T^A)}{g(T^*)}=\frac{1}{2}\left[\frac{T^*}{T^A}+\frac{T^A}{T^*}\right]$$

Which Way to round?

If you need to choose between $Q_1<Q^*$, $Q_2>Q^*$

Choose the one that has smaller geometric deviation.

$\frac{Q^*}{Q_1}$ or $\frac{Q_2}{Q_1}$ // Midpoint is where the two are equal)

$$\frac{Q_m}{Q_1}=\frac{Q_2}{Q_m}\Rightarrow Q_m=\sqrt{Q_1Q_2}$$

Geometric Average

EOQ Power of Two

Orders can only be placed once a unit time

Allowable reorder intervals are T^{PO2} (1, 2, 4, 8, 16, 32, ...)

Quantity Discounts

Interval - Quantities for which the unit cost is unchanged

m - Number of intervals (numbered $0, 1, \ldots, m-1$)

C_J - Unit cost for interval J ($C_0 > C_1 > C_2 > C_{m-1}$)

b_J - Breakpoint J - Beginning of interval J

$$0 = b_0 < b_1 < b_2 < b_{m-1} < b_m = \infty$$

$G_J(Q)$ - average annual cost as a function of Q given that Q falls within interval J.

Plan of attack

- Look at each interval seperately
  Find $Q^{(j)EOQ}$, the unconstrained optimal value for interval J. [Use $Q^{(j)EOQ}$ to find $Q^{(j)*}$]

$b_j \le Q^{(j)EOQ} \le b_{j+1} \Rightarrow Q^{(j)*} = Q^{(j)EOQ}$

$Q^{(j)EOQ} \le b_j \le b_{j+1} \Rightarrow Q^{(j)*} = b_j$

$b_j \le b_{j+1} \le Q^{(j)EOQ} \Rightarrow Q^{(j)*} = b_{j+1}$

$\frac{C(Q)}{Q^*}$

- Find the optimal order quantity by comparing the optimal value for the different intervals.

Quantity Discounts-Holding costs

$h = IC + W$ $\quad \frac{C(Q)}{Q}$ Depends on Q $\Rightarrow$ Holding cost rate is a function of Q.

$$h(Q) = I \cdot \frac{C(Q)}{Q} + W$$

Define: $h_j = IC_J + W$

For All units Discount:

In each interval J find $Q^{(j)EOQ}$

$$G_J(Q) = \frac{C(Q) + K + h(Q) \cdot \frac{1}{2} \cdot T \cdot Q}{T} = \qquad T = \frac{Q}{\lambda}$$

$$= \frac{C(Q)}{Q} \cdot \lambda + \frac{K\lambda}{Q} + \frac{[I \cdot C(Q) + W] \cdot Q}{2}$$

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$$= C_J \lambda + \frac{K\lambda}{Q} + \frac{[IC_J + W]Q}{2} =$$

$$\frac{C(Q)}{Q}$$

$$= C_J \lambda + \frac{K\lambda}{Q} + \frac{h_J Q}{2} \Rightarrow Q^{(j)EOQ} = \sqrt{\frac{2K\lambda}{h_j}}$$

Incremental Discount

For each interval j find $Q^{(j)EOQ}$

$$G_j(Q) = \frac{C(Q) + K + h(Q)\cdot\frac{1}{2}J\cdot Q}{J} =$$

$$= \frac{[C(b_j) + C_j\cdot(Q - b_j)]}{Q}\cdot\lambda + \frac{K\lambda}{Q} +$$

$$+ \frac{\left[I\cdot\frac{C(b_j) + C_j(Q-b_j)}{Q} + W\right]\cdot Q}{2} =$$

$$= C_j\lambda + \frac{[C(b_j) - C_j b_j]}{Q}\lambda + \frac{K\lambda}{Q} + \frac{I[C(b_j) + C_j(Q-b_j)] + WQ}{2} =$$

$$= C_j\lambda + \frac{I[C(b_j) - C_j b_j]}{2} + \frac{[K + C(b_j) - C_j b_j]\lambda}{Q} + \frac{h_j Q}{2}$$

$$\Rightarrow Q^{(j)EOQ} = \sqrt{\frac{2\lambda(K + C(b_j) - C_j b_j)}{h_j}}$$

Deterministic Demand Discrete Time

n - number of periods in the planning horizon

r_i - requirements (demand) in period i (λ_i)

y_i - amount to order (lot size) in period i (Q_i)

I_i - inventory held from period i to period $i+1$

I_0 - initial inventory $(h_0 = 0)$

T - number of periods covered with a certain order (Silver-Meal)

Network Flow

- In each period i units can become available from
  - → Production in period i
  - → Inventory from period $i-1$

- In each period i, units can be used for
  - → Demand in period i
  - → Inventory for period $i+1$

Total Production $\sum_1^n y_i$ must equal total demand $\sum_1^n r_i$

$I_0 = 0$

$I_n = 0$

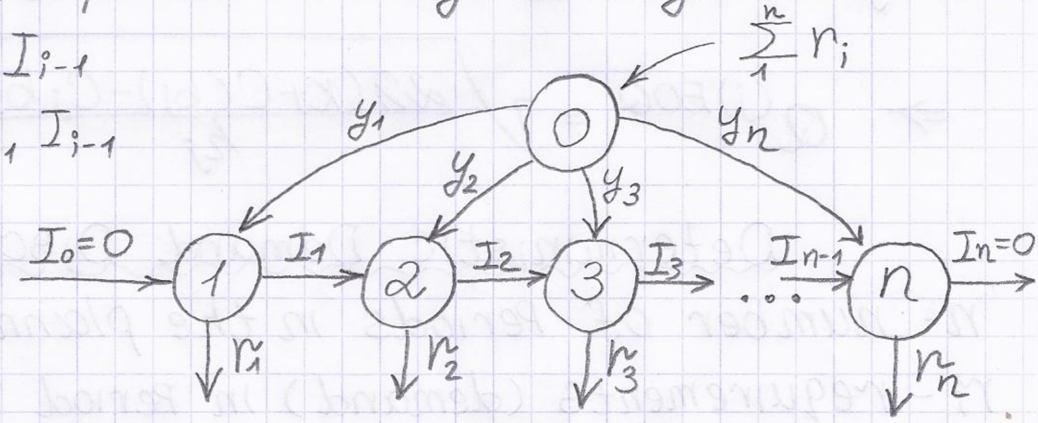
Representation as a Network Flow

Node 0 ⟺ Total production
Flow in is $\sum_1^n r_i$

Node i represents period i
Flow out is r_i

Arc $(0,i)$ represents production in period i
Flow in is I_i
Cost is $(0 \text{ or } K_i + C_i y_i)$

Arc $(i-1,i)$ represents holding inventory from period $i-1$
Flow in is I_{i-1}
Cost is $h_{i-1} I_{i-1}$



MIP Representation

$$\delta_i = \begin{cases} 1 & \text{if } y_i > 0 \\ 0 & \text{if } y_i = 0 \end{cases}$$ indicator variable indicating if production takes place in period i

$$\text{Min} \sum_1^n (K_i \delta_i + C_i y_i + h_{i-1} I_{i-1})$$

s.t. $\sum_1^n r_i = \sum_1^n y_i$ $\qquad$ $I_i, y_i \geq 0$

$I_{i-1} + y_i = r_i + I_i$ $\qquad$ $\delta \in \{0,1\}$

$I_0 = I_n = 0$ $\qquad$ $M = \sum r_i$

$y_i \leq M \delta_i$

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Solution Methods

- Heuristics
  - → Lot for Lot (LFL)
  - → Silver-Meal
- Optimal algorithm
  - → Wagner Whiten (WW)
  - → Zangwill (Include Shortages)

All Satisfy Zero ordering

Lot for Lot

Produce each Period exactly the requirement for that Period

Minimizes inventory costs ↔ Maximizes fixed order costs

Silver-Meal

Start at Period 1 by examining an order for one Period

→ Calculate the average holding and setup cost per Period $C(J)$ if ordered for J Periods

Variable order cost is constant → not considered

→ Increase the order one Period and recalculate

→ Stop when average cost per Period begins to increase

→ Order enough to cover demand for $J-1$ Periods

$$C(J) = \frac{1}{J} \cdot [K + h \cdot r_2 + 2h \cdot r_3 + \ldots + (J-1) h \cdot r_J]$$

if $C(J-1) > C(J)$ then $J = J+1 \ldots$

Set $y_1 = r_1 + \ldots + r_{J-1}$ and $y_2 = \ldots = y_{J-1} = 0$

Wagner Whiten (Optimal Algorithm)

In Period 1 produce enough to last until Period j $(r_1 + \ldots + r_{j-1})$

j goes from 2 to $n+1$

Period $n+1$ signifies the end of the world

The cost for each Path is: $C_{ij} = K_i + C_i \sum_{m=i}^{j-1} r_m + \sum_{m=i+1}^{j-1} \left[r_m \sum_{s=i}^{m-1} h_s \right]$

$$= K_i + C_i \sum_{m=i}^{J-1} r_m + \sum_{S=i}^{J-2} \left[h_S \sum_{m=S+1}^{J-1} r_m \right]$$

Dynamic Programming

f_K - min cost of starting period K with zero inventory and supplying all requirements until the end of the world.

$$f_K = \min_{K+1 \le J \le n+1} (C_{KJ} + f_J)$$

$f_{n+1} = 0$ (Initial condition)

Zangwill

$$C_i(y_i) = K_i \cdot \delta_i + c_i y_i$$

→ Shortages are allowed $C_{iJ} = C_i\left(\sum_{m=i}^{J-1} r_m\right) + \sum_{S=i}^{J-2} H_S \left(\sum_{m=S+1}^{J-1} r_m\right)$

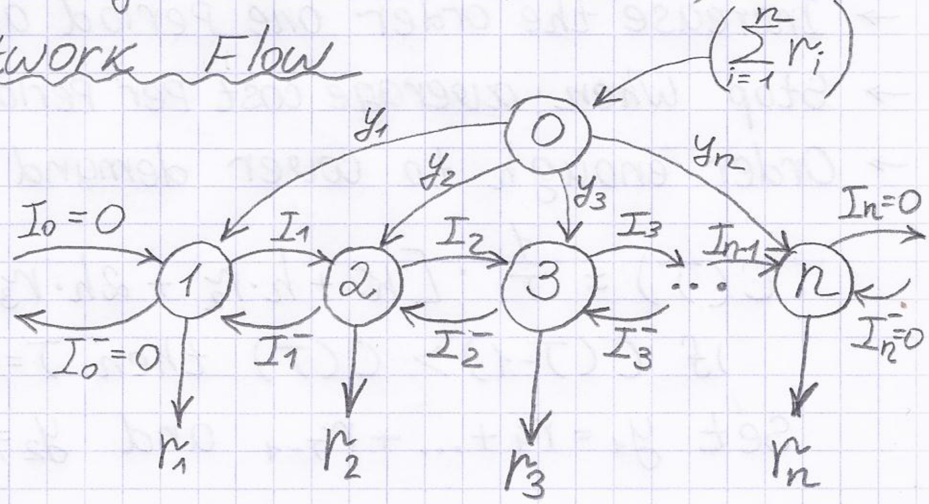
→ I_i^- ⇝ amount of shortage (unsatisfied demand)

$I_i I_i^- = 0$

$S_i(I_i^-)$ - cost of having I_i^- units of unsatisfied demand at the end of period i

→ All shortages satisfied by the end of time (n)

Network Flow



MIP Representation

$$\min \sum_{i=1}^{n} \left(C_i(y_i) + H_i(I_i) + S_i(I_i^-) \right)$$

s.t. $\sum_{i=1}^{n} r_i = \sum_{i=1}^{n} y_i$ $\qquad I_0^- = I_n^- = 0$

$I_{i-1} + y_i + I_i^- = r_i + I_i + I_{i-1}^-$ $\qquad I_i, y_i, I_i^- \ge 0$

$I_0 = I_n = 0$

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Stochastic Demand Discrete Time

One Period model

V - selling Price per unit

C - Purchase cost per unit

h - holding cost per unit (-1 · salvage value)

P - shortage cost per unit (above lost sale)

D - continuous random variable representing demand with known distribution.

d - instance/realization of D

Q - order quantity

C_o - overage cost per unit $= C + h \leftarrow Prob(D<Q) = F(Q)$

C_u - underage cost per unit $= (V-C) + P \leftarrow Prob(D>Q) = 1-F(Q)$

$$C_u \cdot (1-F(Q^*)) = C_o \cdot F(Q^*) \Rightarrow F(Q^*) = \frac{C_u}{(C_u+C_o)} \leftarrow \text{Critical Ratio}$$

$G(Q,d)$ - Overag and underage cost at the end of the Period when Q units are ordered and the demand is d.

$$G(Q) = E_D[G(Q,d)]$$

$$G(Q,d) = C_o(Q-d)^+ + C_u(d-Q)^+$$

$$G(Q) = E_D[G(Q,d)] = \int_0^\infty G(Q,d) \cdot f(d)\,dd$$

$$G(Q) = C_o \int_0^\infty (Q-d)^+ \cdot f(d)\,dd + C_u \int_0^\infty (d-Q)^+ \cdot f(d)\,dd =$$

$$= C_o \int_0^Q (Q-d) \cdot f(d)\,dd + C_u \int_Q^\infty (d-Q) \cdot f(d)\,dd$$

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Service

α - no stock-out probability

β - Fill rate ~ percentage of demand filled off-the-shelf

Evaluation of Shortage cost P using α

- No Stock-out Probability set to α by management

  Given V, c, h and α calculate P.

$$\alpha = F(Q^*) = \frac{C_u}{C_u + C_o} = \frac{P+V-c}{(P+V-c)+(c+h)} = \frac{P+V-c}{P+V+h}$$

$$\alpha \cdot (P+V+h) = P+V-c$$

$$\alpha(V+h) - (V-c) = P(1-\alpha)$$

$$P = \frac{\alpha h + c - (1-\alpha)V}{1-\alpha} = \frac{\alpha h + c}{1-\alpha} - V$$

Initial Inventory

u - starting inventory

Q - order quantity

S - order-up-to quantity

$$S = Q + u$$

S^* - desired order-up-to quantity

S^* is independent of u

S^* = optimal order quantity without Initial inventory

$G(S)$ - Expected cost (overage and underage) above the right decision when ordering-up-to S units

$J(S)$ - Expected Profit when ordering-up-to S units

$$F(S^*) = \frac{C_u}{(C_u + C_o)}$$

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Fixed Order Cost (s, S)

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Indifference Point: $J(u) = J(S^*) - K$

$$(V-C)\mu + C \cdot u - G(u) = (V-C)\mu + C u - G(S^*) - K$$

$$G(u) = G(S^*) + K$$

Define s^* as the smallest initial inventory u such that

$$G(s^*) - G(S^*) = K$$

When the net inventory at the start of the period is less than or equal to s, order to bring the inventory up to S.

Infinite horizon (S)

n - number of Periods

d_i - instance/realization of D in period i

S - Order-up-to level in each Period

h - holding cost Per unit in stock at the end of the period

→ No Salvage value because the world does not end

p - Shortage cost per unit (loss of good will)

→ All shortages satisfied the very next Period

Critical Ratio

Backorders: $F(S^*) = \dfrac{C_u}{C_u + C_o} = \dfrac{p}{p+h}$

Lost sales: $F(S^*) = \dfrac{C_u}{C_u + C_o} = \dfrac{V - C + p}{V - C + p + h}$

Backorders

Expected Profit over n periods $= -C \cdot S + (V-C)(n-1)\mu + V \cdot E[\min(S, D_n)] - n \cdot L(S)$

$$L(S) = h\int_0^S (S-d) f(d)\, dd + p\int_S^\infty (d-S) f(d)\, dd$$

Let n grow without bound → Expected Profit Per Period over infinite horizon ← Divide by n and get Profit per Period

Over infinite horizon, Expected Profit Per Period is

$(V-C)\mu - L(S)$ ← Profit of right decision - cost above the right decision.

$F(S^*) = \frac{p}{p+h} = \frac{C_u}{C_u+C_o}$

→ c may affect h

→ V may affect p

Lost Sales

Expected Profit over n Periods is:

$$-C\cdot S + [n\cdot V - (n-1)c]\left[\mu - \int_S^\infty (d-S)f(d)\,dd\right] - nL(S)$$

Divide by n → expected Profit per Period

$n \to \infty \Rightarrow$ over infinite horizon

over infinite horizon: $(V-C)\left[\mu - \int_S^\infty (d-S)f(d)\,dd\right] - L(S)$

$$F(S^*) = \frac{p+V-C}{p+V-C+h} = \frac{C_u}{C_u+C_o}$$

S^* is larger with lost sales than with Backorders.

Stochastic Demand Continuous Time

Base Stock - S

uncertain demand during supplier lead time

H - on-hand inventory (units)

B - Backorders (units)

I - Net inventory ($I = H - B$)

O - on-order inventory

X - Inventory Position ($X = I + O$)

τ - Lead time

D, d - Random variable and instance of demand during lead-time τ; D is continuous: $F(d)$, $f(d)$, μ, σ^2

h - holding cost rate

S - base stock level

$\hat{p}$ - backorder cost rate

G(S) - expected cost per unit time using base stock level S.

$S = X = (I + O)$

- On-order inventory at time t (now) equals demand in the last τ time units $[t-\tau, t] \Leftrightarrow O = d$
- An order is placed together with each demand.
- Orders before time $t-\tau$ have already arrived
- Orders after time $t-\tau$ have not yet arrived.
- Number of orders between time $t-\tau$ and t is exactly D.

$S = I + d \Rightarrow I = S - d \Rightarrow H - B = S - d \Rightarrow H = S - d + B$

$H = (S-d)^+$

$B = (d-S)^+$

$E[I] = S - \mu$

$E[H] = S - \mu + E[B]$

Optimal Base Stock Policy

$$G(S) = h \cdot E[H] + \hat{p} \cdot E[B]$$

$$E[H] = \int_0^S (S-d) f(d)\,dd + \int_S^\infty 0 \cdot f(d)\,dd$$

$$E[B] = \int_0^S 0 \cdot f(d)\,dd + \int_S^\infty (d-S) f(d)\,dd$$

$$G(S) = h \cdot \int_0^S (S-d) f(d)\,dd + \hat{p} \int_S^\infty (d-S) f(d)$$

(Leibniz's rule)

$$\frac{\partial G(S)}{\partial S} = h \cdot \int_0^S f(d)\,dd - \hat{p} \int_S^\infty f(d)\,dd$$

$$h \cdot F(S^*) - \hat{p}(1 - F(S^*)) = 0 \Rightarrow F(S^*) = \frac{\hat{p}}{h + \hat{p}}$$

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Lot Size / Reorder Point (Q,R)

- When the inventory level reaches R (reorder point) order Q (order quantity) units.

- Shortage cost ≫ holding cost

τ - Lead time

D - Random variable and an instance of demand during the lead time τ.

λ - Expected demand rate $\rightarrow \mu = \lambda \cdot \tau$

S - Safety stock $(S = R - E(D) = R - \lambda\tau)$

K - Fixed ordering cost

C - cost per unit

h - holding cost rate

P - backorder cost per unit

R - reorder point (units)

Q - Lot size (units)

J - average time between orders $J = \frac{Q}{\lambda}$

G(Q,R) - average cost per unit time if using Q and R

Inventory holding cost (Approx 1)

- Inventory level varies 'linearly' between S and S+Q.

$\rightarrow S = E[R-D] = R - \lambda\tau$

Expected holding cost per unit time (Approx 1) is:

$$\frac{h \cdot \left(\frac{S+(S+Q)}{2}\right) \cdot J}{J} = h \cdot \left(\frac{Q}{2} + S\right) = h \cdot \left(\frac{Q}{2} + R - \lambda\tau\right)$$

Expected fixed cost per unit time depends on Q

$$\frac{K}{J} = \frac{K\lambda}{Q}$$

Expected number of stock-outs in a cycle: $n(R)$

$$n(R) = E[(D-R)^+] = \int_R^{\infty} (d-R) f(d)\, dd$$

- Shortage cost for a cycle is independent of Q: $P \cdot n(R)$
- Expected shortage cost per unit time depends on Q:

$$\frac{P \cdot n(R)}{T} = \frac{P \cdot n(R) \cdot \lambda}{Q}$$

- Expected cost per unit time (holding, setup and shortage)

$$G(Q,R) = h\left(\frac{Q}{2} + R - \lambda\tau\right) + K \cdot \frac{\lambda}{Q} + \frac{P \cdot \lambda \cdot n(R)}{Q}$$

$$\frac{\partial G}{\partial Q} = \frac{\partial G}{\partial R} = 0$$

$$Q^* = \sqrt{\frac{2\lambda[K + P \cdot n(R^*)]}{h}} \quad (a)$$

$$F(R^*) = \frac{P\lambda - hQ^*}{P \cdot \lambda} \quad (b)$$

Solution Procedure (Approx 1)

Iterating between equations (a) and (b) until two successive values of Q and R are the 'same'

$$Q^{new} = \sqrt{\frac{2K\lambda}{h}} \quad (a) \quad n(R) = 0$$

$$F(R^{new}) = \frac{P\lambda - h \cdot Q^{new}}{P \cdot \lambda} \Rightarrow R^{new} \quad (b)$$

$Q^{old} = Q^{new}$; $R^{old} = R^{new}$ repeat

$$Q^{new} = \sqrt{\frac{2\lambda[K + P \cdot n(R^{old})]}{h}} \quad (a)$$

$$F(R^{new}) = \frac{P\lambda - h \cdot Q^{new}}{P \cdot \lambda} \Rightarrow R^{new} \quad (b)$$

until $Q^{old} \approx Q^{new}$, $R^{old} \approx R^{new}$

$Q^* = Q^{new}$; $R^* = R^{new}$

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(APPROX 2)

- Inventory level varies 'linearly' between
  → The expected on-hand inventory before order arrives
  $$E[(R-D)^+] = E[R-D+(D-R)^+] = R-\lambda\tau+n(R)$$
  → The expected net inventory after order arrives
  $$S+Q = R-\lambda\tau+Q$$

Expected average holding cost (APPROX 2) is:

$$\frac{h\cdot\left(\frac{(S+n(R))+(Q+S)}{2}\right)\cdot T}{T} = h\left(\frac{Q}{2}+S+\frac{n(R)}{2}\right)$$

- For the same Q and R, approx 2 reports a larger Safety stock than approx 1.

$$Q^* = \sqrt{\frac{2\lambda[K+P\cdot n(R^*)]}{h}} \quad (a')$$

$$F(R^*) = \frac{P\cdot\lambda - \frac{Q^*\cdot h}{2}}{P\cdot\lambda + \frac{Q^*\cdot h}{2}} \quad (b')$$

Same iterations algorithms using a' and b' instead of a and b

- If demand follows a normal distribution $N(\mu,\sigma^2)$
  → $D \sim N(\tau\cdot\mu, \tau\cdot\sigma^2)$

Solution Procedure with Normal Distribution

If $D\sim N(\mu,\sigma^2)$

n(R) is computed using the standartized loss function

$$L(z) = \int_z^\infty (t-z)\phi(t)\,dt$$

$$n(R) = \sigma\cdot L\left(\frac{R-\mu}{\sigma}\right)$$

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Multiple Items Single Stage

EOQ With Constrains

n - number of items

$C_i, K_i, h_i, \lambda_i, Q_i, T_i, G_i(Q_i)$ - Like EOQ $(i = 1, \dots, n)$

W_i - space consumed by unit i

W - total space available

C - total budget available for inventory investment

Q_i^{EOQ} - Order quantity as determined by the EOQ formula

$$Q_i^{EOQ} = \sqrt{\frac{2K_i\lambda_i}{h_i}}$$

Solution Approach

- Solve the problem with help of Lagrangian Multipliers.
  - θ^W - Lagrange multiplier for the space constraint
  - θ^C - Lagrange multiplier for the budget constraint

Space Constraint

$$\min \sum_{i=1}^{n} G_i(Q_i) = \sum_{i=1}^{n} \left(\frac{h_i Q_i}{2} + \frac{K_i \lambda_i}{Q_i} \right)$$

Subject to: $\sum_1^n W_i Q_i \le W$

$Q_i \geq 0$

If $\sum_1^n W_i Q_i^{EOQ} \le W$, then $Q_i^* = Q_i^{EOQ}$ and go no further.

If $\sum_1^n W_i Q_i^{EOQ} > W$, then some Q_i^{EOQ}'s must be reduced.

Lagrange Multiplier

- Call the dual variable for the constraint θ^W
- θ^{W*} is the decrease in the average cost that would result from adding an additional unit of resource

marginal Benefit

Prepared by Boris Milner

interpret θ^w in 3 different ways:

→ How much willing to pay to expand the warehouse

→ Charge the parts for using our space

→ Charge ourselves for violating the constraint

The new target function

find θ^w and $Q_1, Q_2, \dots, Q_n$ to solve the problem:

$$\max_{\theta^w \geq 0} \left\{ \min_{Q_1,\dots,Q_n \geq 0} \left[\sum_1^n \left(\frac{h_i Q_i}{2} + \frac{K_i \lambda_i}{Q_i} \right) + \theta^w \left(\sum W_i Q_i - W \right) \right] \right\}$$

Differentiate and set equal to zero

$$\frac{\partial G}{\partial Q_i} = 0 \quad \text{for } i = 1, \dots, n \quad \text{and} \quad \frac{\partial G}{\partial \theta^{w*}} = 0$$

$$\frac{\partial G}{\partial Q_i} = \frac{h_i}{2} - \frac{K_i \lambda_i}{Q_i^{*2}} + \theta^{w*} w_i = 0 \quad \forall\, i = 1, \dots, n$$

$$\frac{\partial G}{\partial \theta^w} = \sum_{i=1}^n w_i Q_i - W = 0$$

Solving for Q_i^*: $\quad Q_i^* = \sqrt{\dfrac{2K_i \lambda_i}{h_i + 2\theta^{w*} w_i}} \quad i = 1, \dots, n$

$$\sum_1^n w_i Q_i = W$$

Budget constraints

$$Q_i^* = \sqrt{\frac{2K_i \lambda_i}{h_i + 2\theta^{c*} C_i}} \quad \forall\, i = 1, \dots, n \qquad h_i = I \cdot C_i$$

$$\sum C_i \cdot Q_i^* = C$$

$$Q_i = \sqrt{\frac{2K_i \lambda_i}{h_i + 2\theta^c C_i}} = \sqrt{\frac{2K_i \lambda_i}{h_i}} \cdot \sqrt{\frac{1}{1 + 2\theta^c / I}}$$

$$= Q_i^{EOQ} \sqrt{\frac{1}{1 + 2\theta^c / I}} = Q_i^{EOQ} \cdot m$$

$$m = \sqrt{\frac{1}{1 + 2\theta^c / I}}$$

EOQ with common reorder interval

$$Q_j^{EOQ} = \sqrt{\frac{2K_j\lambda_j}{h_j}} = \sqrt{\frac{2K_j\lambda_j}{h_j(1-\lambda_j/P_j)}}$$

$T_j^{EOQ} = \dfrac{Q_j^{EOQ}}{\lambda_j}$; T - The common reorder interval for all products

$g(T)$ - Average total cost as a function of T

Objective: Find a production schedule that produces all items on the single machine in such a way as to satisfy all demand, have no shortages and minimize average cost.

Average Annual Cost

Average cost for product $j \to G_j(Q_j) = \dfrac{K_j\lambda_j}{Q_j} + \dfrac{h'_j Q_j}{2}$

for all products $\sum_{j=1}^{n} G_j(Q_j) = \sum_{j=1}^{n}\left(\dfrac{K_j\lambda_j}{Q_j} + \dfrac{h'_j Q_j}{2}\right)$

$$Q_j = \lambda_j \cdot T \Rightarrow g(T) = \sum_{j=1}^{n} g_j(T) = \sum_{j=1}^{n}\left(\frac{K_j}{T} + \frac{h'_j\lambda_j T}{2}\right)$$

$$g(T) = \frac{\sum_1^n K_j}{T} + \frac{\left(\sum_1^n h'_j \cdot \lambda_j\right)\cdot T}{2}$$

$$T^* = \sqrt{\frac{2\sum_1^n K_j}{\sum_1^n h'_j\lambda_j}}$$

Adding Set-up times

S_j - Setup time for item j

- Total setup time in a cycle, $\sum_{j=1}^{n} S_j$ is independent of T
- Time available for setup is $T - T\cdot\sum_{j=1}^{n}\dfrac{\lambda_j}{P_j} = T\left(1-\sum_1^n \dfrac{\lambda_j}{P_j}\right)$
  - $\to$ If T is small $\rightsquigarrow$ infeasible
  - $\to$ If T is large $\rightsquigarrow$ feasible (Provided ther is some idle time)
- A particular T is feasible if and only if

$$\sum_{j=1}^{n} S_j \le T\left(1-\sum_{j=1}^{n}\frac{\lambda_j}{P_j}\right) \Leftrightarrow \sum_{j=1}^{n}\frac{S_j}{T} + \sum_{j=1}^{n}\left(\frac{\lambda_j}{P_j}\right) \le 1$$

this leads to the constraint

$$T \geq \frac{\sum_j S_j}{1 - \sum_j \left(\frac{\lambda_j}{P_j}\right)} \equiv T_{min}$$

Demand Rate (Forecasting)

- $F_{t-\tau,t}$ - Forecast made in period $t-\tau$ for period t
- $F_t = F_{t-1,t}$
- Time of the last observation $t-1$
- D_t - Observed values of demand (time series) for period t
  - At time t (now) the following information is available $D_t, D_{t-1}, D_{t-2}, \ldots$
- Time series forecast is obtained by applying some set of weights to past data:

$$F_{t+1} = \sum_{n=0}^{\infty} \alpha_n \cdot D_{t-n} \quad \text{for some set of weights } a_0, a_1, \ldots$$

- $e_1, e_2, e_3, \ldots, e_n$ - Forecast errors over past n periods
  Difference between forecast and reality
  - for one step ahead forecasts: $e_t = F_t - D_t = F_{t-1,t} - D_t$
  - for multiple-step-ahead forecasts: $e_t = F_{t-\tau,t} - D_t$

Measures of Forecast Accuracy

- Mean Absolute Deviation (MAD): $MAD = \frac{1}{n}\sum_{t=1}^{n}|e_t|$
- Mean Square Error (MSE): $MSE = \frac{1}{n}\sum_{t=1}^{n} e_t^2$
- Mean Absolute Percentage Error (MAPE): $MAPE = \frac{100}{n} \cdot \sum_{t=1}^{n}\left|\frac{e_t}{D_t}\right|$

A good forecast should be unbiased: $E(e_i) = 0$