## Scattering Theory

**Radial Equation** $\left[-\frac{\hbar^2}{2m}\frac{d^2}{dr^2}+\frac{l(l+1)\hbar^2}{2mr^2}+V(r)\right]u(r)=E\,u(r)$

*Boundary condition* $u(0)=0$

**Solution for free particle** $\Psi=\sqrt{\frac{2k^2}{\pi}}j_l(kr)Y_{lm}(\theta,\phi)$

**Particle current** $\vec{j}\equiv\frac{\hbar}{2mi}\left(\psi^*\nabla\psi-\psi\nabla\psi^*\right)=\frac{\hbar}{m}\mathrm{Im}\left(\psi^*\nabla\psi\right)$

**Bessel function** $j_l(x)=(-1)^l x^l\left(\frac{1}{x}\frac{d}{dx}\right)^l\frac{\sin x}{x}$

$\lim_{x\to\infty}j_l(x)=\frac{1}{x}\sin\left(x-l\frac{\pi}{2}\right)$

$\lim_{x\to 0}j_l(x)=\frac{1}{(2l+1)!!}x^l$

**Free Wave expansion** $e^{ikz}=\sum_{l=0}^{\infty}i^l(2l+1)j_l(kr)P_l(\cos\theta)$

**Partial Wave appr.** $\lim_{r\to\infty}u_{kl}(r)=\sin\left(kr-l\frac{\pi}{2}+\delta_l\right)$

*Limiting condition* $\sqrt{l(l+1)}>kr_0$

where $r_0$ is the potential effective distance

**Scattering Amp.** $f_k(\theta)=\frac{1}{k}\sum_{l=0}^{\infty}(2l+1)e^{i\delta_l}\sin\delta_l\cdot P_l(\cos\theta)$

**Differential Cross section** $d\sigma/d\Omega=\left|f(\theta)\right|^2$

**Total Cross section** $\sigma_{tot}=\frac{4\pi}{k^2}\sum_{l=0}^{\infty}(2l+1)\sin^2\delta_l$

**Born Approximation** $f(\theta)=-\frac{m}{2\pi\hbar^2}\int V(\vec{r})e^{-i\vec{q}\cdot\vec{r}}d^3r$

where $\vec{q}=\vec{k}_f-\vec{k}_i$ $\quad|\vec{q}|=2k\sin\frac{\theta}{2}$

*Central Pot.* $f(\theta)=-\frac{2m}{q\hbar^2}\int_0^{\infty}r\cdot V(r)\sin(qr)\,dr$

condition $\frac{m}{k\hbar^2}\left|\int_0^{\infty}V(r)\left(e^{2ikr}-1\right)dr\right|\ll 1$

## Useful Relations

$Y_{lm}(\pi-\theta,\phi+\pi)=(-1)^l\,Y_{lm}(\theta,\phi)$

$Y_{lm}(\pi-\theta,\phi)=(-1)^{l+m}\,Y_{lm}(\theta,\phi)$

$Y_{l,-m}=(-1)^m\,Y_{lm}^*$

$Y_{lm}(\theta,\phi+\pi)=(-1)^m\,Y_{lm}(\theta,\phi)$

$\left\langle klm\,|\,\frac{1}{r}\,|\,klm\right\rangle=\frac{1}{a_0n^2}$ $\qquad\left\langle klm\,|\,\frac{1}{r^2}\,|\,klm\right\rangle=\frac{1}{a_0^2n^3\left(l+\frac{1}{2}\right)}$

$\left\langle klm\,|\,\frac{1}{r^3}\,|\,klm\right\rangle=\frac{1}{a_0^3n^3l\left(l+\frac{1}{2}\right)(l+1)}$

$Y_{00}=\frac{1}{\sqrt{4\pi}}$ $\qquad Y_{20}=\sqrt{\frac{5}{4\pi}}\left(\frac{3}{2}\cos^2\theta-\frac{1}{2}\right)$

$Y_{11}=-\sqrt{\frac{3}{8\pi}}\sin\theta e^{i\phi}$ $\qquad Y_{21}=-\sqrt{\frac{15}{8\pi}}\sin\theta\cos\theta e^{i\phi}$

$Y_{10}=\sqrt{\frac{3}{4\pi}}\cos\theta$ $\qquad Y_{22}=\frac{1}{4}\sqrt{\frac{15}{2\pi}}\sin^2\theta e^{2i\phi}$

## Time-Independent Perturbation Theory

**Perturbation** $\hat{H}=\hat{H}_0+\hat{W}$ $\qquad W\ll H_0$

while $\hat{H}_0|\varphi_n\rangle=E_n^0|\varphi_n\rangle$ $\qquad\hat{H}|\psi_n\rangle=E_n|\psi_n\rangle$

**First Order** $E_n=E_n^0+\langle\varphi_n|\hat{W}|\varphi_n\rangle$

$|\psi_n\rangle=N\left[|\varphi_n\rangle+\sum_{k\neq n}\frac{\langle\varphi_k|\hat{W}|\varphi_n\rangle}{E_n^0-E_k^0}|\varphi_k\rangle\right]$

**Second Order** $E_n=E_n^0+\langle\varphi_n|\hat{W}|\varphi_n\rangle+\sum_{k\neq n}\frac{\left|\langle\varphi_k|\hat{W}|\varphi_n\rangle\right|^2}{E_n^{\,0}-E_k^{\,0}}$

**Degenerate states** diagonalize the perturbation in each state's degeneracy subspaces, one by one. If the Operator of the degeneracy commutes with the perturbation than the perturbation is diagonal & Perturbation theory gives exact results.

## Time Dependent Perturbation Theory

**Transition probability** $P_{fi}=\frac{1}{\hbar^2}\left|\int_0^t V_{fi}(t')e^{i\omega_{fi}t'}dt'\right|^2$

where $\omega_{fi}=\frac{1}{\hbar}\left(E_f-E_i\right)$ $\quad V_{fi}=\left\langle\varphi_f\,|\,V(\vec{r},t)\,|\,\varphi_i\right\rangle$

*Conditions* $\left|V_{fi}\right|\ll\left|E_f-E_i\right|$ **I-order:** $P_{fi}\ll 1$

**Adiabatic Theorem** short perturbations are felt like delta functions, while slowly changing perturbation will not follow with transition.

**Sinusoidal Perturbation** $P_{fi}\cong\frac{\left|V_{fi}\right|^2}{4\hbar^2}\frac{\sin^2\left(\frac{\omega-|\omega_{fi}|}{2}t\right)}{\left(\frac{\omega-|\omega_{fi}|}{2}\right)^2}$

*Conditions* $t\gg\frac{1}{|\omega_{fi}|}$ $\qquad\left|V_{fi}\right|t\ll\hbar$

**Fermi's Golden Rule** $R_{fi}=\frac{2\pi}{\hbar}\left|V_{fi}\right|^2\rho\left(E_f\right)$

where $\rho\left(E_f\right)$ is energy density of final state

**Atomic Transitions**

*Electric Dipole* $V_{DE}=-\frac{e\varepsilon}{m\omega}p_z\sin\omega t$

*Magnetic Dipole* $V_{DM}=-\frac{e\varepsilon}{2mc}\left(L_x+2S_x\right)\cos\omega t$

*Electric Quadrupole* $V_{QE}=-\frac{e\varepsilon}{2mc}\left(yp_z+zp_y\right)\cos\omega t$

**Selection Rules**

The Integral $\int Y_{l_1m_1}^*Y_{l_2m_2}Y_{l_3m_3}d\Omega\neq 0$ only if

1) $m_1=m_2+m_3$

2) triangle can be created from $l_1,l_2,l_3$

3) parity: $l_1+l_2-l_3=even$

**Useful Relations** for field polarization calculus

$x=-\frac{1}{2}\sqrt{\frac{8\pi}{3}}\left[Y_{11}-Y_{1-1}\right]r$ $\qquad y=-\frac{1}{2i}\sqrt{\frac{8\pi}{3}}\left[Y_{11}+Y_{1-1}\right]r$

$z=\sqrt{\frac{4\pi}{3}}r\cdot Y_{10}$ $\qquad\vec{p}=\frac{mi}{\hbar}\left[H,\vec{r}\right]$

## Angular Momentum

**Rotation Operator** $\hat{R}_{\hat{n}}(\alpha)=\exp\left(-\frac{i}{\hbar}\alpha\vec{L}\cdot\hat{n}\right)$

**Orbital Angular Momentum** $\vec{L}=\vec{r}\times\vec{p}$

$$L_x=yp_z-p_yz \quad L_y=zp_x-p_zx \quad L_z=xp_y-p_xy$$

$$L_x=\frac{\hbar}{i}\left[-\sin\varphi\frac{\partial}{\partial\theta}-\frac{\cos\varphi}{\tan\theta}\frac{\partial}{\partial\varphi}\right] \quad L_y=\frac{\hbar}{i}\left[\cos\varphi\frac{\partial}{\partial\theta}-\frac{\sin\varphi}{\tan\theta}\frac{\partial}{\partial\varphi}\right]$$

$$L_z=\frac{\hbar}{i}\frac{\partial}{\partial\varphi} \qquad L^2=-\hbar^2\left[\frac{\partial^2}{\partial\theta^2}+\frac{1}{\tan\theta}\frac{\partial}{\partial\theta}+\frac{1}{\sin^2\theta}\frac{\partial^2}{\partial^2\varphi}\right]$$

**Spin operator** $\vec{S}=\frac{\hbar}{2}\vec{\sigma}$

**Pauli matrices** $\sigma_x=\begin{pmatrix}0&1\\1&0\end{pmatrix}$ $\sigma_y=\begin{pmatrix}0&-i\\i&0\end{pmatrix}$ $\sigma_z=\begin{pmatrix}1&0\\0&-1\end{pmatrix}$

*prop.* $\left[\sigma_i,\sigma_j\right]=2i\varepsilon_{ijk}\sigma_k \qquad \left\{\sigma_i,\sigma_j\right\}=2\delta_{ij}$

$$\sigma_i\sigma_j=\delta_{ij}+i\varepsilon_{ijk}\sigma_k$$

*Rotation. prop.* $e^{-i\frac{\alpha}{2}\vec{\sigma}\cdot\hat{n}}=\cos\frac{\alpha}{2}-i\vec{\sigma}\cdot\hat{n}\sin\frac{\alpha}{2}$

**Ladder Operators** $J_\pm=J_x\pm iJ_y$

$$J^2=J_-J_++J_z{}^2+\hbar J_z$$

$$J_\pm\left|j,m\right\rangle=\hbar\sqrt{j(j+1)-m(m\pm1)}\left|j,m\pm1\right\rangle$$

**Commutation relations**

$$\left[J_i,J_j\right]=i\hbar\cdot\varepsilon_{ijk}J_k \qquad \left[\vec{J},J^2\right]=0$$

$$\left[J_z,J_\pm\right]=\pm\hbar J_\pm \qquad \left[J^2,J_\pm\right]=0$$

**Spin addition** $\vec{J}=\vec{J}_1+\vec{J}_2$

$$J=\left|j_1-j_2\right|\ldots\left(j_1+j_2\right) \qquad M=m_1+m_2$$

*relations* $J^2=J_1^2+J_2^2+2\vec{J}_1\vec{J}_2$

$$J^2=J_1^2+J_2^2+2J_{1z}J_{2z}+J_{1+}J_{2-}+J_{1-}J_{2+}$$

$$\vec{J}_1\cdot\vec{J}_2=J_{1z}J_{2z}+\frac{1}{2}\left(J_{1+}J_{2-}+J_{1-}J_{2+}\right)$$

**Spin states representation**

$$\left|l+\tfrac{1}{2},m\right\rangle=\tfrac{1}{\sqrt{2l+1}}\left[\sqrt{l+m+\tfrac{1}{2}}\left|m-\tfrac{1}{2},\tfrac{1}{2}\right\rangle+\sqrt{l-m+\tfrac{1}{2}}\left|m+\tfrac{1}{2},-\tfrac{1}{2}\right\rangle\right]$$

$$\left|l-\tfrac{1}{2},m\right\rangle=\tfrac{1}{\sqrt{2l+1}}\left[\sqrt{l+m+\tfrac{1}{2}}\left|m+\tfrac{1}{2},-\tfrac{1}{2}\right\rangle-\sqrt{l-m+\tfrac{1}{2}}\left|m-\tfrac{1}{2},\tfrac{1}{2}\right\rangle\right]$$

$$\Psi_{k,l+\frac{1}{2},m}=\frac{1}{\sqrt{2l+1}}R_{kl}(r)\begin{cases}\sqrt{l+m+\frac{1}{2}}\;Y_{l,m-\frac{1}{2}}\\ \sqrt{l-m+\frac{1}{2}}\;Y_{l,m+\frac{1}{2}}\end{cases}$$

**Spinors**

$$\Psi_{k,l+\frac{1}{2},m}=\frac{1}{\sqrt{2l+1}}R_{kl}(r)\begin{cases}-\sqrt{l-m+\frac{1}{2}}\;Y_{l,m-\frac{1}{2}}\\ \sqrt{l+m+\frac{1}{2}}\;Y_{l,m+\frac{1}{2}}\end{cases}$$

## Hydrogen Atom

**Fine structure constant** $\alpha=\frac{e^2}{\hbar c}\cong\frac{1}{137}$

**Bohr radius** $a_0=\frac{\hbar^2}{\mu e^2}=\frac{\hbar}{mc\alpha}$

**Energy levels** $E_n=-\frac{1}{2}mc^2\alpha^2\frac{1}{n^2}$

**Radial Functions** $R(r)=N\cdot r^l e^{-\frac{r}{na_0}}P_{n,l}(r)$

$$R_{1,0}=2a_0^{-3/2}e^{-\frac{r}{a_0}} \qquad R_{2,0}=2\left(2a_0\right)^{-3/2}\left(1-\frac{r}{2a_0}\right)e^{-\frac{r}{2a_0}}$$

$$R_{2,1}=\frac{1}{\sqrt{3}}\left(2a_0\right)^{-3/2}\frac{r}{a_0}e^{-\frac{r}{2a_0}} \qquad R_{n,n-1}=Cr^{n-1}e^{-\frac{r}{na_0}}$$

## Interaction Hamiltonians + Corrections

**Spin-Orbit Coupling** $H_{SO}=\dfrac{e}{2m^2c^2}\vec{L}\cdot\vec{S}\dfrac{1}{r}\dfrac{\partial V(r)}{\partial r}$

*Hydrogen* $H_{SO}=\dfrac{e^2}{2\mu^2c^2}\vec{L}\cdot\vec{S}\dfrac{1}{r^3}$

*Correction* $\Delta E_{nl}=\dfrac{1}{4}mc^2\dfrac{\alpha^4}{n^3}\begin{cases}\frac{1}{j\left(j+\frac{1}{2}\right)} & j=l+\frac{1}{2}\\ \frac{-1}{\left(j+\frac{1}{2}\right)(j+1)} & j=l-\frac{1}{2}\end{cases}$

**Weakly Relativistic correction**

$$E_K\cong\frac{p^2}{2m}-\frac{p^4}{8m^3c^2} \qquad H_{mv}=-\frac{1}{2mc^2}\left(H_0-V\right)^2$$

*Correction* $\Delta E_{nl}=\dfrac{1}{4}mc^2\dfrac{\alpha^4}{n^3}\left[\dfrac{3}{2n}-\dfrac{2}{l+\frac{1}{2}}\right]$

**Electromagnetic interaction** $H_{EM}=\frac{1}{2m}\left(\vec{p}-\frac{q}{c}\vec{A}\right)^2+q\varphi$

*first order* $H_B=-\frac{q\vec{B}}{2mc}\left(\vec{L}+2\vec{S}\right)=\omega_L\left(\vec{L}+2\vec{S}\right)\cdot\hat{B}$

*Larmor frequency* $\omega_L=-\frac{qB}{2mc}$

*Correction* $\Delta E_l=M_J\omega_L\hbar\left(1\pm\frac{1}{2l+1}\right) \quad J=l\pm\frac{1}{2}$

for weak fields $H_B\ll H_{SO}$

## Identical Particles

**Permutation Operator** $P_{21}\left|1\varphi_1;2\varphi_2\right\rangle=\left|1\varphi_2;2\varphi_1\right\rangle$

*prop.* $P_{21}^\dagger=P_{21} \quad P_{21}^2=1$ eigenvalues: $\pm1$

**Tensor multiplication** $\left\langle1a;2b\,|\,1c;2d\right\rangle=\left\langle1a\,|\,1c\right\rangle\left\langle2b\,|\,2d\right\rangle$

**Symmetrizer** $\hat{S}=\dfrac{1}{N!}\sum_\alpha P_\alpha$

*two particles* $\hat{S}=\frac{1}{\sqrt{2}}\left(1+P_{21}\right)$ (normalized)

**Anti-Symmetrizer**

$$\hat{A}=\frac{1}{N!}\sum_\alpha\varepsilon_\alpha P_\alpha \qquad \varepsilon_\alpha=\begin{cases}1 & \text{even permutation}\\ -1 & \text{odd permutation}\end{cases}$$

*two particles* $\hat{A}=\frac{1}{\sqrt{2}}\left(1-P_{21}\right)$ (normalized)

**Proprieties** $S^\dagger=S^2=S \quad A^\dagger=A^2=A \quad AS=SA=0$

**Symmetrization postulate** a physical system of identical particles can be either completely symmetric or completely anti-symmetric.

# 32. CLEBSCH-GORDAN COEFFICIENTS, SPHERICAL HARMONICS, AND $d$ FUNCTIONS

Note: A square-root sign is to be understood over *every* coefficient, *e.g.*, for −8/15 read $-\sqrt{8/15}$.

Notation:

| | | |
|---|---|---|
| | $J$ $J$ ... | |
| | $M$ $M$ ... | |
| $m_1$ | $m_2$ | Coefficients |
| $m_1$ | $m_2$ | |
| . | . | |

Each table below gives $m_1$, $m_2$ and the coefficients; column headings give $J$ ($M$).

**1/2×1/2**

| $m_1$ | $m_2$ | 1 (+1) |
|---|---|---|
| +1/2 | +1/2 | 1 |

| $m_1$ | $m_2$ | 1 (0) | 0 (0) |
|---|---|---|---|
| +1/2 | −1/2 | 1/2 | 1/2 |
| −1/2 | +1/2 | 1/2 | −1/2 |

| $m_1$ | $m_2$ | 1 (−1) |
|---|---|---|
| −1/2 | −1/2 | 1 |

**1×1/2**

| $m_1$ | $m_2$ | 3/2 (+3/2) |
|---|---|---|
| +1 | +1/2 | 1 |

| $m_1$ | $m_2$ | 3/2 (+1/2) | 1/2 (+1/2) |
|---|---|---|---|
| +1 | −1/2 | 1/3 | 2/3 |
| 0 | +1/2 | 2/3 | −1/3 |

| $m_1$ | $m_2$ | 3/2 (−1/2) | 1/2 (−1/2) |
|---|---|---|---|
| 0 | −1/2 | 2/3 | 1/3 |
| −1 | +1/2 | 1/3 | −2/3 |

| $m_1$ | $m_2$ | 3/2 (−3/2) |
|---|---|---|
| −1 | −1/2 | 1 |

**2×1**

| $m_1$ | $m_2$ | 3 (+3) |
|---|---|---|
| +2 | +1 | 1 |

| $m_1$ | $m_2$ | 3 (+2) | 2 (+2) |
|---|---|---|---|
| +2 | 0 | 1/3 | 2/3 |
| +1 | +1 | 2/3 | −1/3 |

| $m_1$ | $m_2$ | 3 (+1) | 2 (+1) | 1 (+1) |
|---|---|---|---|---|
| +2 | −1 | 1/15 | 1/3 | 3/5 |
| +1 | 0 | 8/15 | 1/6 | −3/10 |
| 0 | +1 | 6/15 | −1/2 | 1/10 |

| $m_1$ | $m_2$ | 3 (0) | 2 (0) | 1 (0) |
|---|---|---|---|---|
| +1 | −1 | 1/5 | 1/2 | 3/10 |
| 0 | 0 | 3/5 | 0 | −2/5 |
| −1 | +1 | 1/5 | −1/2 | 3/10 |

| $m_1$ | $m_2$ | 3 (−1) | 2 (−1) | 1 (−1) |
|---|---|---|---|---|
| 0 | −1 | 6/15 | 1/2 | 1/10 |
| −1 | 0 | 8/15 | −1/6 | −3/10 |
| −2 | +1 | 1/15 | −1/3 | 3/5 |

| $m_1$ | $m_2$ | 3 (−2) | 2 (−2) |
|---|---|---|---|
| −1 | −1 | 2/3 | 1/3 |
| −2 | 0 | 1/3 | −2/3 |

| $m_1$ | $m_2$ | 3 (−3) |
|---|---|---|
| −2 | −1 | 1 |

**1×1**

| $m_1$ | $m_2$ | 2 (+2) |
|---|---|---|
| +1 | +1 | 1 |

| $m_1$ | $m_2$ | 2 (+1) | 1 (+1) |
|---|---|---|---|
| +1 | 0 | 1/2 | 1/2 |
| 0 | +1 | 1/2 | −1/2 |

| $m_1$ | $m_2$ | 2 (0) | 1 (0) | 0 (0) |
|---|---|---|---|---|
| +1 | −1 | 1/6 | 1/2 | 1/3 |
| 0 | 0 | 2/3 | 0 | −1/3 |
| −1 | +1 | 1/6 | −1/2 | 1/3 |

| $m_1$ | $m_2$ | 2 (−1) | 1 (−1) |
|---|---|---|---|
| 0 | −1 | 1/2 | 1/2 |
| −1 | 0 | 1/2 | −1/2 |

| $m_1$ | $m_2$ | 2 (−2) |
|---|---|---|
| −1 | −1 | 1 |

$$Y_1^0 = \sqrt{\frac{3}{4\pi}}\cos\theta$$

$$Y_1^1 = -\sqrt{\frac{3}{8\pi}}\sin\theta\, e^{i\phi}$$

$$Y_2^0 = \sqrt{\frac{5}{4\pi}}\Big(\frac{3}{2}\cos^2\theta - \frac{1}{2}\Big)$$

$$Y_2^1 = -\sqrt{\frac{15}{8\pi}}\sin\theta\cos\theta\, e^{i\phi}$$

$$Y_2^2 = \frac{1}{4}\sqrt{\frac{15}{2\pi}}\sin^2\theta\, e^{2i\phi}$$

$$Y_\ell^{-m} = (-1)^m Y_\ell^{m*}$$

$$d^{\,\ell}_{m,0} = \sqrt{\frac{4\pi}{2\ell+1}}\, Y_\ell^m\, e^{-im\phi}$$

**2×1/2**

| $m_1$ | $m_2$ | 5/2 (+5/2) |
|---|---|---|
| +2 | 1/2 | 1 |

| $m_1$ | $m_2$ | 5/2 (+3/2) | 3/2 (+3/2) |
|---|---|---|---|
| +2 | −1/2 | 1/5 | 4/5 |
| +1 | +1/2 | 4/5 | −1/5 |

| $m_1$ | $m_2$ | 5/2 (+1/2) | 3/2 (+1/2) |
|---|---|---|---|
| +1 | −1/2 | 2/5 | 3/5 |
| 0 | +1/2 | 3/5 | −2/5 |

| $m_1$ | $m_2$ | 5/2 (−1/2) | 3/2 (−1/2) |
|---|---|---|---|
| 0 | −1/2 | 3/5 | 2/5 |
| −1 | +1/2 | 2/5 | −3/5 |

| $m_1$ | $m_2$ | 5/2 (−3/2) | 3/2 (−3/2) |
|---|---|---|---|
| −1 | −1/2 | 4/5 | 1/5 |
| −2 | +1/2 | 1/5 | −4/5 |

| $m_1$ | $m_2$ | 5/2 (−5/2) |
|---|---|---|
| −2 | −1/2 | 1 |

**3/2×1/2**

| $m_1$ | $m_2$ | 2 (+2) |
|---|---|---|
| +3/2 | +1/2 | 1 |

| $m_1$ | $m_2$ | 2 (+1) | 1 (+1) |
|---|---|---|---|
| +3/2 | −1/2 | 1/4 | 3/4 |
| +1/2 | +1/2 | 3/4 | −1/4 |

| $m_1$ | $m_2$ | 2 (0) | 1 (0) |
|---|---|---|---|
| +1/2 | −1/2 | 1/2 | 1/2 |
| −1/2 | +1/2 | 1/2 | −1/2 |

| $m_1$ | $m_2$ | 2 (−1) | 1 (−1) |
|---|---|---|---|
| −1/2 | −1/2 | 3/4 | 1/4 |
| −3/2 | +1/2 | 1/4 | −3/4 |

| $m_1$ | $m_2$ | 2 (−2) |
|---|---|---|
| −3/2 | −1/2 | 1 |

**3/2×1**

| $m_1$ | $m_2$ | 5/2 (+5/2) |
|---|---|---|
| +3/2 | +1 | 1 |

| $m_1$ | $m_2$ | 5/2 (+3/2) | 3/2 (+3/2) |
|---|---|---|---|
| +3/2 | 0 | 2/5 | 3/5 |
| +1/2 | +1 | 3/5 | −2/5 |

| $m_1$ | $m_2$ | 5/2 (+1/2) | 3/2 (+1/2) | 1/2 (+1/2) |
|---|---|---|---|---|
| +3/2 | −1 | 1/10 | 2/5 | 1/2 |
| +1/2 | 0 | 3/5 | 1/15 | −1/3 |
| −1/2 | +1 | 3/10 | −8/15 | 1/6 |

| $m_1$ | $m_2$ | 5/2 (−1/2) | 3/2 (−1/2) | 1/2 (−1/2) |
|---|---|---|---|---|
| +1/2 | −1 | 3/10 | 8/15 | 1/6 |
| −1/2 | 0 | 3/5 | −1/15 | −1/3 |
| −3/2 | +1 | 1/10 | −2/5 | 1/2 |

| $m_1$ | $m_2$ | 5/2 (−3/2) | 3/2 (−3/2) |
|---|---|---|---|
| −1/2 | −1 | 3/5 | 2/5 |
| −3/2 | 0 | 2/5 | −3/5 |

| $m_1$ | $m_2$ | 5/2 (−5/2) |
|---|---|---|
| −3/2 | −1 | 1 |

$$\langle j_1 j_2 m_1 m_2 | j_1 j_2 J M\rangle = (-1)^{J-j_1-j_2}\langle j_2 j_1 m_2 m_1 | j_2 j_1 J M\rangle$$

$$d^{\,j}_{m',m} = (-1)^{m-m'} d^{\,j}_{m,m'} = d^{\,j}_{-m,-m'}$$

**3/2×3/2**

| $m_1$ | $m_2$ | 3 (+3) |
|---|---|---|
| +3/2 | +3/2 | 1 |

| $m_1$ | $m_2$ | 3 (+2) | 2 (+2) |
|---|---|---|---|
| +3/2 | +1/2 | 1/2 | 1/2 |
| +1/2 | +3/2 | 1/2 | −1/2 |

| $m_1$ | $m_2$ | 3 (+1) | 2 (+1) | 1 (+1) |
|---|---|---|---|---|
| +3/2 | −1/2 | 1/5 | 1/2 | 3/10 |
| +1/2 | +1/2 | 3/5 | 0 | −2/5 |
| −1/2 | +3/2 | 1/5 | −1/2 | 3/10 |

| $m_1$ | $m_2$ | 3 (0) | 2 (0) | 1 (0) | 0 (0) |
|---|---|---|---|---|---|
| +3/2 | −3/2 | 1/20 | 1/4 | 9/20 | 1/4 |
| +1/2 | −1/2 | 9/20 | 1/4 | −1/20 | −1/4 |
| −1/2 | +1/2 | 9/20 | −1/4 | −1/20 | 1/4 |
| −3/2 | +3/2 | 1/20 | −1/4 | 9/20 | −1/4 |

| $m_1$ | $m_2$ | 3 (−1) | 2 (−1) | 1 (−1) |
|---|---|---|---|---|
| +1/2 | −3/2 | 1/5 | 1/2 | 3/10 |
| −1/2 | −1/2 | 3/5 | 0 | −2/5 |
| −3/2 | +1/2 | 1/5 | −1/2 | 3/10 |

| $m_1$ | $m_2$ | 3 (−2) | 2 (−2) |
|---|---|---|---|
| −1/2 | −3/2 | 1/2 | 1/2 |
| −3/2 | −1/2 | 1/2 | −1/2 |

| $m_1$ | $m_2$ | 3 (−3) |
|---|---|---|
| −3/2 | −3/2 | 1 |

$$d^{\,1}_{0,0} = \cos\theta \qquad d^{\,1/2}_{1/2,1/2} = \cos\frac{\theta}{2} \qquad d^{\,1}_{1,1} = \frac{1+\cos\theta}{2}$$

$$d^{\,1/2}_{1/2,-1/2} = -\sin\frac{\theta}{2} \qquad d^{\,1}_{1,0} = -\frac{\sin\theta}{\sqrt{2}}$$

$$d^{\,1}_{1,-1} = \frac{1-\cos\theta}{2}$$

**2×3/2**

| $m_1$ | $m_2$ | 7/2 (+7/2) |
|---|---|---|
| +2 | +3/2 | 1 |

| $m_1$ | $m_2$ | 7/2 (+5/2) | 5/2 (+5/2) |
|---|---|---|---|
| +2 | +1/2 | 3/7 | 4/7 |
| +1 | +3/2 | 4/7 | −3/7 |

| $m_1$ | $m_2$ | 7/2 (+3/2) | 5/2 (+3/2) | 3/2 (+3/2) |
|---|---|---|---|---|
| +2 | −1/2 | 1/7 | 16/35 | 2/5 |
| +1 | 1/2 | 4/7 | 1/35 | −2/5 |
| 0 | 3/2 | 2/7 | −18/35 | 1/5 |

| $m_1$ | $m_2$ | 7/2 (+1/2) | 5/2 (+1/2) | 3/2 (+1/2) | 1/2 (+1/2) |
|---|---|---|---|---|---|
| +2 | −3/2 | 1/35 | 6/35 | 2/5 | 2/5 |
| +1 | −1/2 | 12/35 | 5/14 | 0 | −3/10 |
| 0 | 1/2 | 18/35 | −3/35 | −1/5 | 1/5 |
| −1 | 3/2 | 4/35 | −27/70 | 2/5 | −1/10 |

| $m_1$ | $m_2$ | 7/2 (−1/2) | 5/2 (−1/2) | 3/2 (−1/2) | 1/2 (−1/2) |
|---|---|---|---|---|---|
| +1 | −3/2 | 4/35 | 27/70 | 2/5 | 1/10 |
| 0 | −1/2 | 18/35 | 3/35 | −1/5 | −1/5 |
| −1 | 1/2 | 12/35 | −5/14 | 0 | 3/10 |
| −2 | 3/2 | 1/35 | −6/35 | 2/5 | −2/5 |

| $m_1$ | $m_2$ | 7/2 (−3/2) | 5/2 (−3/2) | 3/2 (−3/2) |
|---|---|---|---|---|
| 0 | −3/2 | 2/7 | 18/35 | 1/5 |
| −1 | −1/2 | 4/7 | −1/35 | −2/5 |
| −2 | 1/2 | 1/7 | −16/35 | 2/5 |

| $m_1$ | $m_2$ | 7/2 (−5/2) | 5/2 (−5/2) |
|---|---|---|---|
| −1 | −3/2 | 4/7 | 3/7 |
| −2 | −1/2 | 3/7 | −4/7 |

| $m_1$ | $m_2$ | 7/2 (−7/2) |
|---|---|---|
| −2 | −3/2 | 1 |

**2×2**

| $m_1$ | $m_2$ | 4 (+4) |
|---|---|---|
| +2 | +2 | 1 |

| $m_1$ | $m_2$ | 4 (+3) | 3 (+3) |
|---|---|---|---|
| +2 | +1 | 1/2 | 1/2 |
| +1 | +2 | 1/2 | −1/2 |

| $m_1$ | $m_2$ | 4 (+2) | 3 (+2) | 2 (+2) |
|---|---|---|---|---|
| +2 | 0 | 3/14 | 1/2 | 2/7 |
| +1 | 1 | 4/7 | 0 | −3/7 |
| 0 | 2 | 3/14 | −1/2 | 2/7 |

| $m_1$ | $m_2$ | 4 (+1) | 3 (+1) | 2 (+1) | 1 (+1) |
|---|---|---|---|---|---|
| +2 | −1 | 1/14 | 3/10 | 3/7 | 1/5 |
| +1 | 0 | 3/7 | 1/5 | −1/14 | −3/10 |
| 0 | 1 | 3/7 | −1/5 | −1/14 | 3/10 |
| −1 | 2 | 1/14 | −3/10 | 3/7 | −1/5 |

| $m_1$ | $m_2$ | 4 (0) | 3 (0) | 2 (0) | 1 (0) | 0 (0) |
|---|---|---|---|---|---|---|
| +2 | −2 | 1/70 | 1/10 | 2/7 | 2/5 | 1/5 |
| +1 | −1 | 8/35 | 2/5 | 1/14 | −1/10 | −1/5 |
| 0 | 0 | 18/35 | 0 | −2/7 | 0 | 1/5 |
| −1 | 1 | 8/35 | −2/5 | 1/14 | 1/10 | −1/5 |
| −2 | 2 | 1/70 | −1/10 | 2/7 | −2/5 | 1/5 |

| $m_1$ | $m_2$ | 4 (−1) | 3 (−1) | 2 (−1) | 1 (−1) |
|---|---|---|---|---|---|
| +1 | −2 | 1/14 | 3/10 | 3/7 | 1/5 |
| 0 | −1 | 3/7 | 1/5 | −1/14 | −3/10 |
| −1 | 0 | 3/7 | −1/5 | −1/14 | 3/10 |
| −2 | 1 | 1/14 | −3/10 | 3/7 | −1/5 |

| $m_1$ | $m_2$ | 4 (−2) | 3 (−2) | 2 (−2) |
|---|---|---|---|---|
| 0 | −2 | 3/14 | 1/2 | 2/7 |
| −1 | −1 | 4/7 | 0 | −3/7 |
| −2 | 0 | 3/14 | −1/2 | 2/7 |

| $m_1$ | $m_2$ | 4 (−3) | 3 (−3) |
|---|---|---|---|
| −1 | −2 | 1/2 | 1/2 |
| −2 | −1 | 1/2 | −1/2 |

| $m_1$ | $m_2$ | 4 (−4) |
|---|---|---|
| −2 | −2 | 1 |

$$d^{\,3/2}_{3/2,3/2} = \frac{1+\cos\theta}{2}\cos\frac{\theta}{2}$$

$$d^{\,3/2}_{3/2,1/2} = -\sqrt{3}\,\frac{1+\cos\theta}{2}\sin\frac{\theta}{2} \qquad d^{\,2}_{2,2} = \Big(\frac{1+\cos\theta}{2}\Big)^2$$

$$d^{\,3/2}_{3/2,-1/2} = \sqrt{3}\,\frac{1-\cos\theta}{2}\cos\frac{\theta}{2} \qquad d^{\,2}_{2,1} = -\frac{1+\cos\theta}{2}\sin\theta$$

$$d^{\,3/2}_{3/2,-3/2} = -\frac{1-\cos\theta}{2}\sin\frac{\theta}{2} \qquad d^{\,2}_{2,0} = \frac{\sqrt{6}}{4}\sin^2\theta \qquad d^{\,2}_{1,1} = \frac{1+\cos\theta}{2}(2\cos\theta-1)$$

$$d^{\,3/2}_{1/2,1/2} = \frac{3\cos\theta-1}{2}\cos\frac{\theta}{2} \qquad d^{\,2}_{2,-1} = -\frac{1-\cos\theta}{2}\sin\theta \qquad d^{\,2}_{1,0} = -\sqrt{\frac{3}{2}}\sin\theta\cos\theta$$

$$d^{\,3/2}_{1/2,-1/2} = -\frac{3\cos\theta+1}{2}\sin\frac{\theta}{2} \qquad d^{\,2}_{2,-2} = \Big(\frac{1-\cos\theta}{2}\Big)^2 \qquad d^{\,2}_{1,-1} = \frac{1-\cos\theta}{2}(2\cos\theta+1) \qquad d^{\,2}_{0,0} = \Big(\frac{3}{2}\cos^2\theta - \frac{1}{2}\Big)$$

**Figure 32.1**: The sign convention is that of Wigner (*Group Theory*, Academic Press, New York, 1959), also used by Condon and Shortley (*The Theory of Atomic Spectra*, Cambridge Univ. Press, New York, 1953), Rose (*Elementary Theory of Angular Momentum*, Wiley, New York, 1957), and Cohen (*Tables of the Clebsch-Gordan Coefficients*, North American Rockwell Science Center, Thousand Oaks, Calif., 1974). The coefficients here have been calculated using computer programs written independently by Cohen and at LBNL.

## Convolution

$$f(x) * g(x) = \int dx' \, f(x - x')g(x')$$

$$f * g = g * f$$

$$(f * g) * h = f * (g * h)$$

$$\frac{\partial}{\partial x}(f * g) = \left(\frac{\partial}{\partial x} f\right) * g = f * \left(\frac{\partial}{\partial x} g\right)$$

$$\left(\frac{\partial^n}{\partial x^n}\delta\right) * f = \frac{\partial^n}{\partial x^n} f(0)$$

## Dirac's δ function

$$\int dx \delta(x - x_0) f(x) = f(x_0)$$

$$\int dx \delta^{(n)}(x - x_0) f(x) = (-1)^n f^{(n)}(x_0)$$

$$\delta^3(\mathbf{r} - \mathbf{r_0}) = \delta(x - x_0)\delta(y - y_0)\delta(z - z_0)$$

$$\delta^{(n)}(x - x_0) f(x) = \sum_{k=0}^{n} (-1)^k f^{(k)}(x_0)\delta^{(n-k)}(x - x_0)$$

$$\delta(f(x)) = \frac{1}{|f'(x_0)|}\delta(x - x_0)$$

$$\delta(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ikx} dk$$

## Commutator

$$[A, B] = -[B, A]$$

$$[A, B]^\dagger = [B^\dagger, A^\dagger]$$

$$[AB, C] = A[B, C] + [A, C]B$$

$$[A, BC] = B[A, C] + [A, B]C$$

$$[A,[B,C]] + [B,[C,A]] + [C,[A,B]] = 0 \qquad \text{(Jacobi identity)}$$

If A and B both commute with [A,B], then:

$$\begin{cases} [A, f(B)] = [A,B]f'(B) \qquad [f(A), B] = [A,B]f'(A) \\ e^A e^B = e^{A+B} e^{\frac{1}{2}[A,B]} \qquad \text{(Grauber's Formula)} \end{cases}$$

$$\frac{d}{dt}\langle A \rangle = \frac{i}{\hbar}\langle [H, A] \rangle + \left\langle \frac{\partial A}{\partial t} \right\rangle$$

$$e^A B e^{-A} = B + [A, B] + \frac{1}{2!}[A,[A,B]] + \ldots \qquad \text{(Campbell-Baker-Hausdorff formula)}$$

# 1-D Fourier Transform

$f, \bar{f}, g, \bar{g}$ are functions from $R$ to $C$, $r, r_0, p, p_0$ are real numbers.

| $f(x)$ | $\bar{f}(p)$ |
|---|---|
| $\langle x \vert y \rangle$ | $\langle p \vert y \rangle$ |
| $f(x)$ | $\frac{1}{\sqrt{2p\hbar}} \int dx\, e^{-\frac{i}{\hbar}px} f(x)$ |
| $\frac{1}{\sqrt{2p\hbar}} \int dp\, e^{\frac{i}{\hbar}px} \bar{f}(p)$ | $\bar{f}(p)$ |
| $\frac{\partial f}{\partial x}$ | $\frac{i}{\hbar} p\bar{f}$ |
| $xf$ | $i\hbar \frac{\partial \bar{f}}{\partial p}$ |
| $f(x+x_0)$ | $e^{\frac{i}{\hbar}px_0} \bar{f}(p)$ |
| $e^{\frac{i}{\hbar}px} f(x)$ | $\bar{f}(p-p_0)$ |
| $f(ax)$ | $\frac{1}{\lvert a \rvert} \bar{f}\left(\frac{p}{a}\right)$ |
| $f * g$ | $\sqrt{2p\hbar}\, \bar{f}\bar{g}$ |
| $fg$ | $\frac{1}{\sqrt{2p\hbar}} \bar{f} * \bar{g}$ |
| $\frac{\partial^n}{\partial x^n} d(x)$ | $\frac{1}{\sqrt{2p\hbar}} \left(\frac{i}{\hbar} p\right)^n$ |
| $x^n$ | $\sqrt{2p\hbar}\,(i\hbar)^n \frac{\partial^n}{\partial p^n} d(p)$ |

# 3-D Fourier Transform

$f, \bar{f}, g, \bar{g}$ are functions from $R^3$ to $C$, $\mathbf{r}, \mathbf{r_0}, \mathbf{p}, \mathbf{p_0}$ are vectors.

| $f(\mathbf{r})$ | $\bar{f}(\mathbf{p})$ |
|---|---|
| $\langle \mathbf{r} \vert y \rangle$ | $\langle \mathbf{p} \vert y \rangle$ |
| $f(\mathbf{r})$ | $\frac{1}{(2p\hbar)^{3/2}} \int d^3 r\, e^{-\frac{i}{\hbar}\mathbf{p}\cdot\mathbf{r}} f(\mathbf{r})$ |
| $\frac{1}{(2p\hbar)^{3/2}} \int d^3 p\, e^{\frac{i}{\hbar}\mathbf{p}\cdot\mathbf{r}} \bar{f}(\mathbf{p})$ | $\bar{f}(\mathbf{p})$ |
| $X(x)Y(y)Z(z)$ | $\bar{X}(p_x)\bar{Y}(p_y)\bar{Z}(p_z)$ |
| $\frac{\partial f}{\partial x}$ | $\frac{i}{\hbar} p_x \bar{f}$ |
| $xf$ | $i\hbar \frac{\partial \bar{f}}{\partial p_x}$ |
| $f(\mathbf{r}+\mathbf{r_0})$ | $e^{\frac{i}{\hbar}\mathbf{p}\cdot\mathbf{r_0}} \bar{f}(\mathbf{p})$ |
| $e^{\frac{i}{\hbar}\mathbf{p_0}\cdot\mathbf{r}} f(\mathbf{r})$ | $\bar{f}(\mathbf{p}-\mathbf{p_0})$ |
| $f(a\mathbf{r})$ | $\frac{1}{\vert a \vert^3} \bar{f}\left(\frac{\mathbf{p}}{a}\right)$ |
| $f * g$ | $(2p\hbar)^{3/2} \bar{f}\bar{g}$ |
| $fg$ | $\frac{1}{(2p\hbar)^{3/2}} \bar{f} * \bar{g}$ |
| $\frac{\partial^n}{\partial x^n} d^3(\mathbf{r})$ | $\frac{1}{(2p\hbar)^{3/2}} \left(\frac{i}{\hbar} p_x\right)^n$ |
| $x^n$ | $(2p\hbar)^{3/2} (i\hbar)^n \frac{\partial^n}{\partial p_x^{\ n}} d^3(\mathbf{p})$ |

# x- and p- Representations

$$\langle \mathbf{r} | \mathbf{p} \rangle = \frac{1}{(2p\hbar)^{3/2}} e^{\frac{i}{\hbar}\mathbf{r}\cdot\mathbf{p}}$$

| | |
|---|---|
| $\langle \mathbf{r} \| \mathbf{r'} \rangle = d(\mathbf{r}-\mathbf{r'})$ | $\langle \mathbf{p} \| \mathbf{p'} \rangle = d(\mathbf{p}-\mathbf{p'})$ |
| $f(\hat{\mathbf{r}})\|\mathbf{r}\rangle = f(\mathbf{r})\|\mathbf{r}\rangle$ | $f(\hat{\mathbf{p}})\|\mathbf{p}\rangle = f(\mathbf{p})\|\mathbf{p}\rangle$ |
| $\langle \mathbf{r} \| \hat{p}_x \| \mathbf{r'} \rangle = \frac{\hbar}{i}\frac{\partial}{\partial x} d(\mathbf{r}-\mathbf{r'})$ | $\langle \mathbf{p} \| \hat{x} \| \mathbf{p'} \rangle = i\hbar \frac{\partial}{\partial p_x} d(\mathbf{p}-\mathbf{p'})$ |
| $\langle \mathbf{r} \| \hat{p}_x \| y \rangle = \frac{\hbar}{i}\frac{\partial y(\mathbf{r})}{\partial x}$ | $\langle \mathbf{p} \| \hat{x} \| y \rangle = i\hbar \frac{\partial \bar{y}(\mathbf{p})}{\partial p_x}$ |
| $\langle \mathbf{r} \| \bar{f}(\hat{\mathbf{p}}) \| \mathbf{r'} \rangle = \frac{1}{(2p\hbar)^{3/2}} f(\mathbf{r}-\mathbf{r'})$ | $\langle \mathbf{p} \| f(\hat{\mathbf{r}}) \| \mathbf{p'} \rangle = \frac{1}{(2p\hbar)^{3/2}} \bar{f}(\mathbf{p}-\mathbf{p'})$ |

Virial Theorem: $\langle u_n | \frac{p_i^2}{2m} | u_n \rangle = \frac{1}{2} \langle u_n | x_i \frac{\partial V}{\partial x_i} | u_n \rangle \qquad \left( H = \frac{p^2}{2m} + V \right)$

# The Continuity Equation

$$i\hbar \frac{\partial y}{\partial t} + \frac{\hbar^2}{2m}\nabla^2 y = Vy \;\Rightarrow\; i\hbar \frac{\partial}{\partial t}|y|^2 + \frac{\hbar^2}{2m}\nabla\cdot\left(y^*\nabla y - y\nabla y^*\right) = 0$$

$$\Rightarrow \begin{cases} \frac{\partial}{\partial t}|y|^2 + \nabla\cdot\mathbf{j} = 0 \\ \mathbf{j} \equiv \frac{\hbar}{2mi}\left(y^*\nabla y - y\nabla y^*\right) = \frac{\hbar}{m}\mathrm{Im}\left(y^*\nabla y\right) \end{cases}$$

$$\frac{\partial}{\partial t} P\{\mathbf{r}\in V\} = -\int_{\partial V} \mathbf{j}\cdot\hat{\mathbf{n}}\,ds$$

# Matrix Representation of Operators

$$\hat{T}|e_j\rangle = \sum_i T_{ij}|e_i\rangle$$

$$T_{ij} = \langle e_i|\hat{T}|e_j\rangle$$

$$\hat{T}|v\rangle = \sum_j \hat{T}|e_j\rangle\langle e_j|v\rangle = \sum_j \hat{T}|e_j\rangle v_j = \sum_{i,j} T_{ij}|e_i\rangle v_j$$

$$\langle e_i|\hat{T}|v\rangle = \sum_j T_{ij}v_j$$

# Coupling Between Energy-States

$$H = \begin{pmatrix} E_1 & W_{12} \\ W_{12} & E_2 \end{pmatrix}$$

$$E_{\pm} = \frac{E_1 + E_2}{2} \pm \sqrt{\left(\frac{E_1 - E_2}{2}\right)^2 + W_{12}{}^2}$$

$$\begin{pmatrix} |y_+\rangle \\ |y_-\rangle \end{pmatrix} = \begin{pmatrix} \cos q & \sin q \\ -\sin q & \cos q \end{pmatrix} \begin{pmatrix} |j_1\rangle \\ |j_2\rangle \end{pmatrix}$$

$$\tan(2q) = \frac{2W_{12}}{E_1 + E_2}$$

# Oscillations Between States (Rabi's Formula)

$$|y(0)\rangle = |j_1\rangle = \cos q\, |j_+\rangle - \sin q\, |j_-\rangle$$

$$|y(t)\rangle = \cos q\, e^{-\frac{i}{\hbar}E_+ t} |j_+\rangle - \sin q\, e^{-\frac{i}{\hbar}E_- t} |j_-\rangle$$

$$\left|\langle j_2 | y(t)\rangle\right|^2 = \sin^2(2q) \sin^2\left(\frac{(E_+ - E_-)t}{2\hbar}\right)$$

# 1-D Simple Harmonic Oscillator

$$X = \sqrt{\tfrac{mw}{\hbar}}\, x \quad ; \quad P = \tfrac{1}{\sqrt{m\hbar w}}\, p \quad ; \quad [X, P] = i$$

$$A = \tfrac{1}{\sqrt{2}}(X + iP) \quad ; \quad A^{\dagger} = \tfrac{1}{\sqrt{2}}(X - iP) \quad ; \quad [A, A^{\dagger}] = 1$$

$$H = \frac{p^2}{2m} + \frac{mw^2 x^2}{2} = \frac{\hbar w}{2}(X^2 + P^2) = \hbar w\left(A^{\dagger}A + \tfrac{1}{2}\right) \quad ; \quad [H, A] = -\hbar w A$$

$$A|E_n\rangle = \sqrt{n}\,|E_{n-1}\rangle$$

$$A^{\dagger}|E_n\rangle = \sqrt{n+1}\,|E_{n+1}\rangle$$

$$A^{\dagger}A|E_n\rangle = n|E_n\rangle$$

$$X = \tfrac{1}{\sqrt{2}}(A^{\dagger} + A) \quad ; \quad P = \tfrac{i}{\sqrt{2}}(A^{\dagger} - A)$$

$$x = \sqrt{\tfrac{\hbar}{2mw}}(A^{\dagger} + A) \quad ; \quad p = i\sqrt{\tfrac{m\hbar w}{2}}(A^{\dagger} - A)$$

# Stationary States of 1-D SHO in x- Representation

$$u_n(x) = \langle x | E_n \rangle = \langle x | \tfrac{1}{\sqrt{n!}} \left(A^\dagger\right)^n | E_0 \rangle = \frac{1}{\sqrt{n!}} \left( \sqrt{\frac{\hbar}{2mw}} x - \sqrt{\frac{mw}{2\hbar}} \frac{d}{dx} \right)^n \left[ \sqrt[4]{\left(\frac{mw}{p\hbar}\right)} e^{-\frac{mwx^2}{2\hbar}} \right]$$

$$x = \sqrt{\frac{\hbar}{mw}} x$$

$$u_n(x) = \langle x | E_n \rangle = \frac{1}{\sqrt{2^n n!}} \sqrt[4]{\left(\frac{mw}{p\hbar}\right)} \left( x - \frac{d}{dx} \right)^n e^{-\frac{x^2}{2}}$$

*Schrödinger Equation*: $u_n'' + \left(2n + 1 - x^2\right) u_n = 0$

# Hermite Polynomials

$$u_n(x) = C_n H_n(x) e^{-\frac{x^2}{2}}$$

$$H_n(x) = e^{\frac{x^2}{2}} \left( x - \frac{d}{dx} \right)^n e^{-\frac{x^2}{2}}$$

$$C_n = \frac{1}{\sqrt{2^n n!}} \sqrt[4]{\left(\frac{mw}{p\hbar}\right)}$$

$$\begin{aligned} H_0 &= 1 \\ H_1 &= 2x \\ H_2 &= 4x^2 - 2 \\ H_3 &= 8x^3 - 12x \\ H_4 &= 16x^4 - 48x^2 + 12 \\ H_5 &= 32x^5 - 160x^3 + 120x \end{aligned}$$

Differential Equation: $H_n'' - 2xH_n' + 2nH_n = 0$

Rodrigues Formula: $H_n = (-1)^n e^{x^2} \frac{d^n}{dx^n} e^{-x^2}$

Polynomial Coefficients: $a_{m+2} = \frac{-2(n-m)}{(m+1)(m+2)} a_m$

Orthogonality: $\int_{-\infty}^{\infty} dx\, e^{-x^2} H_m(x) H_n(x) = d_{nm} 2^n n! \sqrt{p}$

Generating Function: $e^{2tx - t^2} = \sum_{n=0}^{\infty} \frac{H_n(x)}{n!} t^n$

Recurrence Formulas:
$$\begin{cases} H_{n+1} = 2xH_n - 2nH_{n-1} \\ H_{n+1} = 2xH_n - H_n' \\ H_n' = 2nH_{n-1} \end{cases}$$

Another thing: $H_{2k}(0) = (-2)^k (2k-1)!!$

# Angular Momentum (J)

$\left[J_i, J_j\right] = i\hbar \cdot e_{ijk} J_k$

$\mathbf{J} \times \mathbf{J} = i\hbar \mathbf{J}$

$\left[\mathbf{J}, J^2\right] = 0$

$J_\pm = J_x \pm i J_y$

$\left[J_z, J_\pm\right] = \pm\hbar J_\pm$

$\left[J^2, J_\pm\right] = 0$

$\left[J_z, J_\pm\right] = \pm\hbar J_\pm$

$J^2 = J_+ J_- + J_z{}^2 - \hbar J_z$
$J^2 = J_- J_+ + J_z{}^2 + \hbar J_z$

<u>Eigenstates</u>:

$$J^2 \left| jm \right\rangle = j(j+1)\hbar^2 \left| jm \right\rangle$$

$$J_z \left| jm \right\rangle = m\hbar \left| jm \right\rangle$$

$$j = 0, \tfrac{1}{2}, 1, \tfrac{3}{2}, 2, \ldots$$
$$m = -j, -j+1, \ldots, j-1, j$$

$J_\pm \left| j, m \right\rangle = \hbar \sqrt{j(j+1) - m(m \pm 1)} \left| j, m \pm 1 \right\rangle$

# Orbital Angular Momentum (L)

$\mathbf{L} = \mathbf{r} \times \mathbf{p}$

$$\mathbf{L} = \frac{\hbar}{i}\left( \hat{\mathbf{f}} \frac{\partial}{\partial q} - \hat{\mathbf{?}} \frac{1}{\sin q} \frac{\partial}{\partial j} \right)$$

$$L_x = \frac{\hbar}{i}\left[ -\sin j \frac{\partial}{\partial q} - \frac{\cos j}{\tan q} \frac{\partial}{\partial j} \right] \qquad L_y = \frac{\hbar}{i}\left[ \cos j \frac{\partial}{\partial q} - \frac{\sin j}{\tan q} \frac{\partial}{\partial j} \right] \qquad L_z = \frac{\hbar}{i} \frac{\partial}{\partial j}$$

$$L_\pm = \hbar e^{\pm i j} \left[ \pm \frac{\partial}{\partial q} + i \cot q \frac{\partial}{\partial j} \right]$$

$$L^2 = \hbar^2 \left[ -r^2 \nabla^2 + \frac{\partial}{\partial r}\left( r^2 \frac{\partial}{\partial r} \right) \right] \qquad L^2 = -\hbar^2 \left[ \frac{\partial^2}{\partial q^2} + \frac{1}{\tan q} \frac{\partial}{\partial q} + \frac{1}{\sin^2 q} \frac{\partial^2}{\partial^2 j} \right]$$

$$L^2 = -\frac{\hbar^2}{\sin^2 q} \left[ \sin q \frac{\partial}{\partial q}\left( \sin q \frac{\partial}{\partial q} \right) + \frac{\partial^2}{\partial^2 j} \right]$$

# Angular Momentum Matrices

## $s = 1/2$

$$\mathbf{S} = \frac{\hbar}{2}\mathbf{s}$$

$$s_x = \begin{pmatrix} & 1 \\ 1 & \end{pmatrix} \qquad s_y = \begin{pmatrix} & -i \\ i & \end{pmatrix} \qquad s_z = \begin{pmatrix} 1 & \\ & -1 \end{pmatrix}$$

$$s_i s_j = d_{ij} + i e_{ijk} s_k$$

$$\left[s_i, s_j\right] = 2ie_{ijk} s_k$$

$$\left\{s_i, s_j\right\} = 2d_{ij}$$

## $l = 1$

$$L_x = \frac{\hbar}{\sqrt{2}}\begin{pmatrix} & 1 & \\ 1 & & 1 \\ & 1 & \end{pmatrix} \qquad L_y = \frac{i\hbar}{\sqrt{2}}\begin{pmatrix} & -1 & \\ 1 & & -1 \\ & 1 & \end{pmatrix}$$

$$L_x^{\ 2} = \frac{\hbar^2}{2}\begin{pmatrix} 1 & & 1 \\ & 2 & \\ 1 & & 1 \end{pmatrix} \qquad L_y^{\ 2} = \frac{\hbar^2}{2}\begin{pmatrix} 1 & & -1 \\ & 2 & \\ -1 & & 1 \end{pmatrix}$$

## $l = 2$

$$L_x = \hbar\begin{pmatrix} & 1 & & 0 & \\ 1 & & \sqrt{3/2} & & 0 \\ & \sqrt{3/2} & & \sqrt{3/2} & \\ 0 & & \sqrt{3/2} & & 1 \\ & 0 & & 1 & \end{pmatrix} \qquad L_y = i\hbar\begin{pmatrix} & -1 & & 0 & \\ 1 & & -\sqrt{3/2} & & 0 \\ & \sqrt{3/2} & & -\sqrt{3/2} & \\ 0 & & \sqrt{3/2} & & -1 \\ & 0 & & 1 & \end{pmatrix}$$

$$L_x^{\ 2} = \hbar^2\begin{pmatrix} 1 & & \sqrt{3/2} & & 0 \\ & 5/2 & & 3/2 & \\ \sqrt{3/2} & & 3 & & \sqrt{3/2} \\ & 3/2 & & 5/2 & \\ 0 & & \sqrt{3/2} & & 1 \end{pmatrix} \qquad L_y^{\ 2} = \hbar^2\begin{pmatrix} 1 & & -\sqrt{3/2} & & 0 \\ & 5/2 & & -3/2 & \\ -\sqrt{3/2} & & 3 & & -\sqrt{3/2} \\ & -3/2 & & 5/2 & \\ 0 & & -\sqrt{3/2} & & 1 \end{pmatrix}$$

# Associated Legendre Polynomials & Spherical Harmonics

$$\langle \mathbf{r} | lm \rangle = f(r) Y_{lm}(q, j)$$

$$Y_{lm}(q, j) = \sqrt{\frac{2l+1}{4p}\frac{(l-m)!}{(l+m)!}} P_{lm}(\cos q) e^{imj}$$

$$P_{lm}(x) = (-1)^m (1-x^2)^{m/2} \frac{d^m}{dx^m} P_l(x)$$

$$P_l(x) = \frac{1}{2^l l!}\frac{d^l}{dx^l}(x^2-1)^l$$

$$P_{mm}(x) = (-1)^m (2m-1)!!(1-x^2)^{m/2}$$

$$\int_{-1}^{1} P_l(x) P_{l'}(x) dx = \frac{2}{2l+1} d_{ll'}$$

$$\int_{4p} Y_{lm}^{*}(q, j) Y_{l'm'}(q, j) d\Omega = d_{ll'} d_{mm'}$$

$$Y_{l,-m}(q, j) = (-1)^m Y_{l,-m}^{*}(q, j)$$

$$Y_{lm}(p - q, p + j) = (-1)^l Y_{lm}(q, j)$$

$$Y_{l,0}(q, j) = \sqrt{\frac{2l+1}{4p}} P_l(\cos q)$$

Probability: $P(l,m) = \left| \langle lm | y \rangle \right|^2 = \int_0^\infty r^2 dr \left| \int_{4p} Y_{lm}^{*}(q, j) y(r, q, j) d\Omega \right|^2$

$P_0 = 1$ $\qquad P_1 = x$ $\qquad P_2 = \frac{1}{2}\left(3x^2 - 1\right)$

$Y_{00} = \frac{1}{\sqrt{4p}}$ $\qquad Y_{10} = \sqrt{\frac{3}{4p}} \cos q$ $\qquad Y_{1\pm1} = \mp\sqrt{\frac{3}{8p}} e^{\pm ij} \sin q$

$Y_{20} = \sqrt{\frac{5}{16p}}\left(3\cos^2 q - 1\right)$ $\qquad Y_{2\pm1} = \mp\sqrt{\frac{15}{8p}} e^{\pm ij} \sin q \cos q$ $\qquad Y_{2\pm2} = \sqrt{\frac{15}{32p}} e^{\pm 2ij} \sin^2 q$

# Central Potential

Hamiltonian in Spherical Coordinates: $H y = \left[ -\frac{\hbar^2}{2mr}\frac{\partial^2}{\partial r^2} r + \frac{L^2}{2mr^2} + V(r) \right] y$

$$H = \frac{p_r{}^2}{2m} + \frac{L^2}{2mr^2} + V(r)$$

$$p_r = \tfrac{1}{2}\left( \frac{\mathbf{r}\cdot\mathbf{p}}{r} + \frac{\mathbf{p}\cdot\mathbf{r}}{r} \right) = \frac{\hbar}{i}\frac{1}{r}\left( \frac{\partial}{\partial r} r \right)$$

Separation of Variables: $y(\mathbf{r}) = \frac{1}{r} u(r) Y_{lm}(q, j)$

$$\int_0^\infty |u(r)|^2 \, dr = 1$$

$$\lim_{r \to 0} u(r) = 0$$

$u(r \to 0) \approx r^{l+1}$ if V is smaller than $\frac{1}{r^2}$.

Radial Schrödinger Equation: $\left[ -\frac{\hbar^2}{2mr}\frac{d^2}{dr^2} + \frac{l(l+1)\hbar^2}{2mr^2} + V(r) \right] u(r) = Eu(r)$

<u>Free Particle</u>:

$$k^2 = \frac{2mE}{\hbar^2}$$

$$r = kr$$

$$\left( \frac{d^2}{dr^2} - \frac{l(l+1)}{r^2} + 1 \right) u = 0 \qquad \text{(Spherical Bessel Equation)}$$

# The Hydrogen Atom

$$V(r) = \frac{e^2}{r^2}$$

$$a_0 = \left(\frac{\hbar^2}{me^2}\right)$$

$$E_n = -\left(\frac{me^4}{2\hbar^2}\right)\frac{1}{n^2} = -\left(\frac{e^2}{2a_0}\right)\frac{1}{n^2} = \frac{mc^2}{2}\cdot a^2 \cdot \frac{1}{n^2}$$

$$R(r) = \frac{1}{r}u(r) = Nr^l L_{n+1}^{2l+1}(r)e^{-\frac{r}{2}} \qquad r \equiv \frac{2r}{na_0}$$

$$R(r) = r^l e^{-\frac{r}{na_0}} P_{n,l}(r)$$

$$R_{1,0} = 2a_0^{-3/2} e^{-\frac{r}{a_0}}$$

$$R_{2,0} = 2(2a_0)^{-3/2}\left(1-\frac{r}{2a_0}\right)e^{-\frac{r}{2a_0}}$$

$$R_{2,1} = \tfrac{1}{\sqrt{3}}(2a_0)^{-3/2}\left(\frac{r}{a_0}\right)e^{-\frac{r}{2a_0}}$$

$\underline{l = n-1}$

$$R_{n,n-1} = Cr^{n-1}e^{-\frac{r}{na_0}} \qquad r_{\max} = a_0 n^2$$

$$< r >= a_0 n\left(n+\tfrac{1}{2}\right) \qquad < r^2 >= a_0{}^2 n^2\left(n+\tfrac{1}{2}\right)(n+1) \qquad \Delta r = \tfrac{1}{2}a_0 n\sqrt{2n+1}$$

# Non-degenerate Time-Independent Perturbation Theory

$$\hat{H} = \hat{H}_0 + \hat{W}$$

$$\hat{H}_0 \left| j_n \right\rangle = E_n^0 \left| j_n \right\rangle \qquad \hat{H} \left| y_n \right\rangle = E_n \left| y_n \right\rangle$$

$$\hat{H}_0 = l \hat{H}_1, \qquad l << 1$$

$$E_n(l) = E_n^0 + \sum_{i=1}^{\infty} l^i E_n^i$$

$$\left| y_n(l) \right\rangle = N(l) \left[ \left| j_n \right\rangle + \sum_{k \neq n} c_{nk}(l) \left| j_k \right\rangle \right]$$

$$c_{nk} = \sum_{i=1}^{\infty} l^i c_{nk}^i$$

<u>First Order</u>:

$$\left| y_n \right\rangle = N \left[ \left| j_n \right\rangle + \sum_{k \neq n} \frac{\left\langle j_k \right| \hat{W} \left| j_n \right\rangle}{E_n^0 - E_k^0} \left| j_k \right\rangle \right]$$

$$E_n = E_n^0 + \left\langle j_n \right| \hat{W} \left| j_n \right\rangle$$

<u>Second Order</u>:

$$E_n = E_n^0 + \left\langle j_n \right| \hat{W} \left| j_n \right\rangle + \sum_{k \neq n} \frac{\left| \left\langle j_k \right| \hat{W} \left| j_n \right\rangle \right|^2}{E_n^0 - E_k^0}$$

$$\sum_{k \neq n} \frac{\left| \left\langle j_k \right| \hat{W} \left| j_n \right\rangle \right|^2}{E_n^0 - E_k^0} \leq \frac{(\Delta W)_n^2}{\Delta E}$$

# Vector Formulas

$$\mathbf{a}\times(\mathbf{b}\times\mathbf{c}) = (\mathbf{a}\cdot\mathbf{c})\mathbf{b} - (\mathbf{a}\cdot\mathbf{b})\mathbf{c}$$

$$(\mathbf{a}\times\mathbf{b})\cdot(\mathbf{c}\times\mathbf{d}) = (\mathbf{a}\cdot\mathbf{c})(\mathbf{b}\cdot\mathbf{d}) - (\mathbf{a}\cdot\mathbf{d})(\mathbf{b}\cdot\mathbf{c})$$

$$\nabla\times(\nabla\times\mathbf{a}) = \nabla(\nabla\cdot\mathbf{a}) - \nabla^2\mathbf{a}$$

$$\nabla\cdot(y\mathbf{a}) = \mathbf{a}\cdot\nabla y + y\nabla\cdot\mathbf{a}$$

$$\nabla\times(y\mathbf{a}) = \nabla y\times\mathbf{a} + y\nabla\times\mathbf{a}$$

$$\nabla(\mathbf{a}\cdot\mathbf{b}) = (\mathbf{a}\cdot\nabla)\mathbf{b} + (\mathbf{b}\cdot\nabla)\mathbf{a} + \mathbf{a}\times(\nabla\times\mathbf{b}) + \mathbf{b}\times(\nabla\times\mathbf{a})$$

$$\nabla\cdot(\mathbf{a}\times\mathbf{b}) = \mathbf{b}\cdot(\nabla\times\mathbf{a}) - \mathbf{a}\cdot(\nabla\times\mathbf{b})$$

$$\nabla\times(\mathbf{a}\times\mathbf{b}) = \mathbf{a}(\nabla\cdot\mathbf{b}) - \mathbf{b}(\nabla\cdot\mathbf{a}) + (\mathbf{b}\cdot\nabla)\mathbf{a} - (\mathbf{a}\cdot\nabla)\mathbf{b}$$

# Vector Calculus Theorems

$$\int_V \nabla\cdot\mathbf{A}\,d^3x = \oint_S \mathbf{A}\cdot\mathbf{n}\,ds$$

$$\int_V \nabla y\,d^3x = \oint_S y\,\mathbf{n}\,ds$$

$$\int_V \nabla\times\mathbf{A}\,d^3x = \oint_S \mathbf{n}\times\mathbf{A}\,ds$$

$$\int_V \left(f\nabla^2 y + \nabla f\cdot\nabla y\right)d^3x = \oint_S f\mathbf{n}\cdot\nabla y\,ds$$

$$\int_V \left(f\nabla^2 y - y\nabla^2 f\right)d^3x = \oint_S \left(f\nabla y - y\nabla f\right)\cdot\mathbf{n}\,ds$$

$$\int_S (\nabla\times\mathbf{A})\cdot\mathbf{n}\,ds = \oint_C \mathbf{A}\cdot d\mathbf{l}$$

$$\int_S \mathbf{n}\times\nabla y\,ds = \oint_C y\,d\mathbf{l}$$

Explicit Forms of Vector Operations (from Jackson)

# Explicit Forms of Vector Operations

Let $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$ be orthogonal unit vectors associated with the coordinate directions specified in the headings on the left, and $A_1$, $A_2$, $A_3$ be the corresponding components of $\mathbf{A}$. Then

**Cartesian** $(x_1, x_2, x_3 = x, y, z)$

$$\nabla\psi = \mathbf{e}_1 \frac{\partial\psi}{\partial x_1} + \mathbf{e}_2 \frac{\partial\psi}{\partial x_2} + \mathbf{e}_3 \frac{\partial\psi}{\partial x_3}$$

$$\nabla\cdot\mathbf{A} = \frac{\partial A_1}{\partial x_1} + \frac{\partial A_2}{\partial x_2} + \frac{\partial A_3}{\partial x_3}$$

$$\nabla\times\mathbf{A} = \mathbf{e}_1\left(\frac{\partial A_3}{\partial x_2} - \frac{\partial A_2}{\partial x_3}\right) + \mathbf{e}_2\left(\frac{\partial A_1}{\partial x_3} - \frac{\partial A_3}{\partial x_1}\right) + \mathbf{e}_3\left(\frac{\partial A_2}{\partial x_1} - \frac{\partial A_1}{\partial x_2}\right)$$

$$\nabla^2\psi = \frac{\partial^2\psi}{\partial {x_1}^2} + \frac{\partial^2\psi}{\partial {x_2}^2} + \frac{\partial^2\psi}{\partial {x_3}^2}$$

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**Cylindrical** $(\rho, \phi, z)$

$$\nabla\psi = \mathbf{e}_1 \frac{\partial\psi}{\partial\rho} + \mathbf{e}_2 \frac{1}{\rho}\frac{\partial\psi}{\partial\phi} + \mathbf{e}_3 \frac{\partial\psi}{\partial z}$$

$$\nabla\cdot\mathbf{A} = \frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho A_1) + \frac{1}{\rho}\frac{\partial A_2}{\partial\phi} + \frac{\partial A_3}{\partial z}$$

$$\nabla\times\mathbf{A} = \mathbf{e}_1\left(\frac{1}{\rho}\frac{\partial A_3}{\partial\phi} - \frac{\partial A_2}{\partial z}\right) + \mathbf{e}_2\left(\frac{\partial A_1}{\partial z} - \frac{\partial A_3}{\partial\rho}\right) + \mathbf{e}_3 \frac{1}{\rho}\left(\frac{\partial}{\partial\rho}(\rho A_2) - \frac{\partial A_1}{\partial\phi}\right)$$

$$\nabla^2\psi = \frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial\psi}{\partial\rho}\right) + \frac{1}{\rho^2}\frac{\partial^2\psi}{\partial\phi^2} + \frac{\partial^2\psi}{\partial z^2}$$

---

**Spherical** $(r, \theta, \phi)$

$$\nabla\psi = \mathbf{e}_1 \frac{\partial\psi}{\partial r} + \mathbf{e}_2 \frac{1}{r}\frac{\partial\psi}{\partial\theta} + \mathbf{e}_3 \frac{1}{r\sin\theta}\frac{\partial\psi}{\partial\phi}$$

$$\nabla\cdot\mathbf{A} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 A_1) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta A_2) + \frac{1}{r\sin\theta}\frac{\partial A_3}{\partial\phi}$$

$$\nabla\times\mathbf{A} = \mathbf{e}_1 \frac{1}{r\sin\theta}\left[\frac{\partial}{\partial\theta}(\sin\theta A_3) - \frac{\partial A_2}{\partial\phi}\right] + \mathbf{e}_2\left[\frac{1}{r\sin\theta}\frac{\partial A_1}{\partial\phi} - \frac{1}{r}\frac{\partial}{\partial r}(rA_3)\right] + \mathbf{e}_3 \frac{1}{r}\left[\frac{\partial}{\partial r}(rA_2) - \frac{\partial A_1}{\partial\theta}\right]$$

$$\nabla^2\psi = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\psi}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\psi}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2\psi}{\partial\phi^2}$$

$$\left[\text{Note that } \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\psi}{\partial r}\right) \equiv \frac{1}{r}\frac{\partial^2}{\partial r^2}(r\psi)\right]$$

# The Postulates of Quantum Mechanics

1. The state of the system is represented by a vector in a Hilbert space.8
2. A physical quantity is represented by an observable (e.g. a Hermitian operator that its eigenvectors form a complete set)
3. The possible outcomes of a measurement of A are only eigenvalues of A.
4. When measuring A in a normalized state $|y\rangle$, the probability of measuring the value "a" is $\sum_i \left|\langle y_a^{(i)} | y \rangle\right|^2$, where $|y_a^{(i)}\rangle$ is an orthonormal basis to the space of A's eigenstates with the eigenvalue "a".
5. After a measurement of A, which yields the value "a", the system is left in an eigenstates of A with the eigenvalue "a".
6. The state's time-development: $i\hbar \frac{d}{dt}|y(t)\rangle = \hat{H}|y(t)\rangle$

   (H is the system's classical Hamiltonian)
7. $[q_i, p_i] = i\hbar d_{ij}$