100 $\quad$ 1 : Re

1) $\int_{-2}^{4} |x|^{4.5} = \int_{-2}^{0} (-x)^{4.5}\, dx + \int_{0}^{4} x^{4.5}\, dx =$

$= \left[\frac{(-x)^{5.5}}{-5.5}\right]_{-2}^{0} + \left[\frac{x^{5.5}}{5.5}\right]_{0}^{4} = \left[0 + \frac{2^{5.5}}{5.5}\right] + \left[\frac{4^{5.5}}{5.5} - 0\right] =$

$= \frac{4^{5.5} + 2^{5.5}}{5.5} = \frac{4^5 \cdot \sqrt{4} + 2^5 \sqrt{2}}{5.5} = \frac{2^{11} + 32\sqrt{2}}{5.5} = \boxed{380.5918}$ ✓

2) $\int_{0}^{1} \frac{5+4x}{3+7x}\, dx = \int_{0}^{1} \frac{5}{3+7x} + \frac{4}{7}\int_{0}^{1} \frac{7x+3-3}{3+7x}\, dx =$

$= \frac{5}{7} \ln(3+7x)\Big|_{0}^{1} + \frac{4}{7}\int_{0}^{1}\left(1 - \frac{3}{3+7x}\right) dx =$

$= \frac{5}{7}\left[\ln(10) - \ln(3)\right] + \frac{4}{7}\left[x - \frac{3}{7}\ln(3+7x)\right]_{0}^{1} =$

$= \frac{5}{7}\ln(10/3) + \frac{4}{7}\left(1 - \frac{3}{7}\ln(10/3)\right) = \frac{23}{49}\ln(10/3) + \frac{4}{7} = \boxed{1.1366}$ ✓

3) $\int_{-\pi}^{\pi} \sin(nx)\sin(mx)\, dx = \frac{1}{2}\int_{-\pi}^{\pi}\Big(\cos((n-m)x) - \cos((n+m)x)\Big)\, dx =$

$n \neq m$ ، اذا $\quad = \frac{1}{2}\left[\frac{\sin((n-m)x)}{(n-m)} - \frac{\sin((n+m)x)}{(n+m)}\right]_{-\pi}^{\pi} = \boxed{0}$ ✓

$n = m$ ، اذا $\quad = \frac{1}{2}\int_{-\pi}^{\pi}(1 - \cos(2nx))\, dx = \frac{1}{2}\left[x - \frac{\sin(2nx)}{2n}\right]_{-\pi}^{\pi} = \frac{1}{2}(2\pi - 0) = \boxed{\pi}$ ✓

4) $\int_{0}^{\infty} e^{-ax^2}\, dx = \frac{1}{2}\sqrt{\frac{\pi}{a}}$

نص:

$\int_{0}^{\infty} e^{-ax^2} \cdot x^2\, dx = \left(-\frac{1}{2a}\right)\int_{0}^{\infty} x(-2ax)\, e^{-ax^2}\, dx =$

$= \left(-\frac{1}{2a}\right)\left\{ x e^{-ax^2}\Big|_{0}^{\infty} - \int_{0}^{\infty} e^{-ax^2} \cdot 1\, dx \right\} = \left(-\frac{1}{2a}\right)\left\{\lim_{x\to\infty} x e^{-ax^2} - 0\right.$

$\left. - \frac{1}{2}\sqrt{\frac{\pi}{a}}\right\} = \left(-\frac{1}{2a}\right)\left[-\frac{1}{2}\sqrt{\frac{\pi}{a}}\right] = \boxed{\frac{1}{4a}\sqrt{\frac{\pi}{a}}}$ ✓

דרך 2

$$\int_0^\infty e^{-x^2}dx = \frac{1}{2}\int_{-\infty}^{\infty} e^{-x^2}dx .$$

$$\left(\int_{-\infty}^{\infty} e^{-x^2}dx\right)^2 = \int_{-\infty}^{\infty} e^{-x^2}dx \int_{-\infty}^{\infty} e^{-y^2}dy = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-(x^2+y^2)}dx\,dy = \circledast$$

$x = r\cos\theta \; ; \; y = r\sin\theta$ מעבר לקואורדינטות קוטביות

$$\text{יעקוביאן} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r\cos^2\theta + r\sin^2\theta = r.$$

$$\circledast = \int_0^{2\pi}\int_0^{\infty} e^{-r^2}\cdot r\,dr\,d\theta = 2\pi\int_0^{\infty} re^{-r^2}dr = 2\pi\left(-e^{-r^2}/2\right)\Big|_0^{\infty} =$$

$$= \frac{2\pi}{2}\left[\lim_{r\to\infty}\left(-e^{-r^2}\right) + 1\right] = \pi .$$

$$\Rightarrow \int_{-\infty}^{\infty} e^{-x^2}dx = \sqrt{\pi} \Rightarrow \int_0^{\infty} e^{-x^2}dx = \boxed{\frac{\sqrt{\pi}}{2}}$$

שאלה 3

א.

1) $f(x) = \sin(x^2)$, $x_0 = \pi$. $f(\pi) = \sin(\pi^2) = -0.43$

$f^{(1)}(x) = \cos(x^2) \cdot 2x$, $f^{(1)}(\pi) = 2\pi\cos(\pi^2) = -5.672$

$f^{(2)}(x) = 2\cos(x^2) - 4x^2\sin(x^2)$, $f^{(2)}(\pi) = 2\cos(\pi^2) - 4\pi^2\sin(\pi^2) = 15.1822$

$f^{(3)}(x) = -4x\sin(x^2) - 8x\sin(x^2) - 8x^3\cos(x^2)$

$f^{(3)}(\pi) = -4\pi\sin(\pi^2) - 8\pi\sin(\pi^2) - 8\pi^3\cos(\pi^2) = 240.1333$

$f^{(4)}(x) = -4\sin(x^2) - 8x^2\cos(x^2) - 8\sin(x^2) - 16x^2\cos(x^2) - 24x^2\cos(x^2) + 16x^4\sin(x^2)$

$\underline{n=4}$

פיתוח טיילור סביב π $f(x) = f(\pi) + \frac{f^{(1)}(\pi)(x-\pi)^1}{1!} + \frac{f^{(2)}(\pi)(x-\pi)^2}{2!} + \frac{f^{(3)}(\pi)(x-\pi)^3}{3!} + \frac{f^{(4)}(\xi(x))(x-\pi)^4}{4!}$

$\Rightarrow f(x) = -0.43 - 5.672(x-\pi) + 7.5911(x-\pi)^2 + 40.022(x-\pi)^3 + \frac{f^{(4)}(\xi(x))(x-\pi)^4}{4!}$

השארית $R(x) = \frac{f^{(4)}(\xi(x))(x-\pi)^4}{4!}$, $\xi(x) \in int(0.1, \pi)$

$|f^{(4)}(x)| \leq |12 + 48x^2 + 16x^4| \underset{x\in(0.1,\pi)}{\leq} 12 + 48\pi^2 + 16\pi^4 = 2044.3 \Rightarrow \boxed{|R_x| \leq \frac{2044.3 \cdot (0.1-\pi)^4}{4!} = 7290.2}$

2) $f(x) = e^{\sin(x)}$, $x_0 = \pi$ $f(\pi) = e^{\sin(\pi)} = e^0 = \underline{1}$

$f'(x) = e^{\sin x} \cdot \cos x$ $f'(\pi) = e^0(-1) = \underline{-1}$

$f^{(2)}(x) = -\sin x e^{\sin x} + \cos^2 x\, e^{\sin x}$; $f^{(2)}(\pi) = 0 + 1 \cdot e^0 = \underline{1}$

$f^{(3)}(x) = -\cos x e^{\sin x} - \sin x\cos x e^{\sin x} - 2\cos x\sin x e^{\sin x} + \cos^3 x\, e^{\sin x} =$

$= (-\cos x - 3\overbrace{\sin x\cos x}^{=\sin(2x)/2} + \cos^3 x)e^{\sin x}$

$f^{(3)}(\pi) = (+1 - 0 - 1) \cdot 1 = 0$

$f^{(4)}(x) = \left(\sin x - \frac{3}{2}(2\cos(2x)) - 3\cos^2 x\sin x\right)e^{\sin x} +$

$+ (-\cos^2 x - 3\sin x\cos^2 x + \cos^3 x)e^{\sin x}$

$f^{(4)}(\pi) = (0 - 3 - 0) \cdot 1 = \underline{-3}$

$f^{(4)}(x) = (\sin x - 3\cos(2x) - 6\cos^2 x\sin x - \cos^2 x + \cos^3 x)e^{\sin x} =$

$= (\sin x - 3\cos(2x) - 3\cos x\sin(2x) - \cos^2 x + \cos^3 x)e^{\sin x}$

$$f^{(5)}(x) = \left(\cos x + 6\sin(2x) + 3\sin x \sin 2x - 6\cos x \cos 2x + \sin 2x - 3\cos^2 x \sin x\right) e^{\sin x}$$
$$+ \left(\tfrac{1}{2}\sin 2x - 3\cos 2x \cdot \cos x - 3\cos^2 x \sin 2x - \cos^3 x + \cos^4 x\right) e^{\sin x}$$

$\underline{n=4}$

$$f(x) \quad 1 + \frac{(-1)(x-\pi)}{1} + \frac{1\cdot(x-\pi)^2}{2!} + \frac{0\cdot(x-\pi)^3}{3!} + \frac{(-3)(x-\pi)^4}{4!} + \frac{f^{(5)}(\xi(x))(x-\pi)^5}{5!} =$$

$$= 1 - (x-\pi)^1 + \frac{1}{2}(x-\pi)^2 - \frac{1}{8}(x-\pi)^4 + \frac{f^{(5)}(\xi(x))(x-\pi)^5}{5!}$$

$$\left|f^{(5)}(x)\right| \le e^{\sin x}\left(1+6+3+6+1+3+\tfrac{1}{2}+3+3+1+1\right) = e^{\sin x}\cdot 28.5$$

$$\le 28.5e = 77.47 \Rightarrow |R_n(x)| \le \frac{77.47(0.1-\pi)^5}{5!} = \underline{55.253}$$

3) $f(x) = \dfrac{1}{(1+x)^{10}}$, $x_0 = \pi$. $\quad f(\pi) = \dfrac{1}{(1+\pi)^{10}}$

$f(x) = (1+x)^{-10}$

$f^{(1)}(x) = -10(1+x)^{-11}$ $\quad f^{(1)}(\pi) = -10(1+\pi)^{-11}$

$f^{(2)}(x) = +10\cdot 11\cdot(1+x)^{-12}$ $\quad f^{(2)}(\pi) = 10\cdot 11(1+\pi)^{-12}$

$f^{(3)}(x) = -10\cdot 11\cdot 12(1+x)^{-13}$, $\quad f^{(3)}(\pi) = -10\cdot 11\cdot 12(1+\pi)^{-13}$

$f^{(4)}(x) = 10\cdot 11\cdot 12\cdot 13(1+x)^{-14}$.

$\underline{n=4}$

$$f(x) = (1+\pi)^{-10} - 10(1+\pi)^{-11}(x-\pi) + 55(1+\pi)^{-12}(x-\pi)^2 - 220(x-\pi)^3 +$$
$$+ \frac{f^{(4)}(\xi)(x-\pi)^4}{4!}, \quad \xi \in \mathrm{int}(x,\pi)$$

$$\left|f^{(4)}(x)\right| \le 10\cdot 11\cdot 12\cdot 13\cdot(1+0.1)^{-14} = 4518.76$$

$$\Rightarrow |R_n(x)| \le \frac{4518.76\cdot(0.1-\pi)^4}{4!} = 16114.33$$

שאלה 4 $\underline{3}$

א.

$y - 2x + z = 4$

$x - y + 3z = 1$

$3x - 4y = -39$

$$\begin{array}{cccc} x & y & z & \\ \end{array}$$

$$\left(\begin{array}{ccc|c} -2 & 1 & 1 & 4 \\ 1 & -1 & 3 & 1 \\ 3 & -4 & 0 & -39 \end{array}\right) \longrightarrow \left(\begin{array}{ccc|c} -2 & 1 & 1 & 4 \\ 0 & -1 & 7 & 6 \\ 0 & -5 & 3 & -66 \end{array}\right) \longrightarrow \left(\begin{array}{ccc|c} -2 & 1 & 1 & 4 \\ 0 & -1 & 7 & 6 \\ 0 & 0 & -32 & -96 \end{array}\right)$$

$\Rightarrow \quad -32z = -96 \Rightarrow \underline{z = 3}$

$-y + 7z = 6 \Rightarrow -y + 21 = 6 \Rightarrow y = 21 - 6 = \underline{15}$

$-2x + y + z = 4 \Rightarrow -2x + 15 + 3 = 4 \Rightarrow x = 7$

$(x, y, z) = (7, 15, 3)$

ב. matlab (מצורף הקוד)

$$A = \begin{bmatrix} -2 & 1 & 1 \\ 1 & -1 & 3 \\ 3 & -4 & 0 \end{bmatrix} \qquad b = \begin{bmatrix} 4 \\ 1 \\ -39 \end{bmatrix}$$

$\underline{A}x = b \Rightarrow x = A^{-1}b$