OPTICS 114210 - Homework Exercises

B. Fourier Theory

1. The functions $g_1(x)$ and $g_2(x)$ are defined in terms of f(x) by convolutions:

$$g_1(x)=[f(x)*f(x)]\cdot f(x)$$

$$g_2(x)=f(x)*[f(x)\cdot f(x)]$$

Find the *difference* $g_1(x)-g_2(x)$ for the following two functions:

$f(x)=\mathrm{rect}(x)$

$f(x)=\delta(x-a)+\delta(x)+\delta(x+a)$, where $\delta(x)$ is the Dirac delta-function.

2. Show that a pair of square pulses can be defined in two ways:

(a) $f(x)= \mathrm{rect}(x/a) * [\delta(x-b) + \delta(x+b)]$

(b) $f(x)= \mathrm{rect}[x/(b+a)] - \mathrm{rect}[x/(b-a)]$

Fourier transform each of the above and show that they are also equal.

3. Find the Fourier transforms of

(a) an exponential decay at $x>0$, i.e. $f(x) = 0$ at $x<0$, $f(x) = \exp(-\alpha x)$ at $x\geq 0$;

(b) the exponentially decaying set of pulses:

$$f(x)=\sum_{m=0}^{\infty}\delta(x-mx_0)\exp(-\alpha m)$$

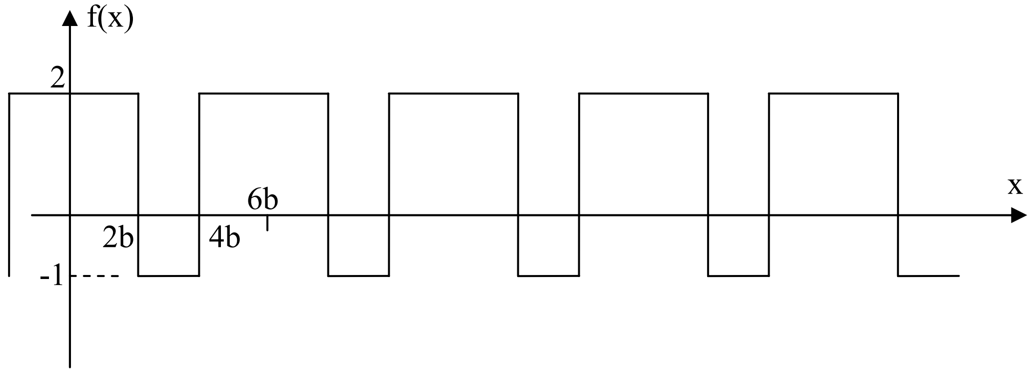
4. A periodic array of δ-functions has every fifth member missing. What is its Fourier transform?

5. (a) Given that f(x, y) has Fourier transform F(u,v), what is the Fourier transform of f(x/a, y/b)?
(b) The function circ(r/R) has transform $2\pi R^2 J_1(\rho R)/\rho R$ where $\rho=\sqrt{(u^2+v^2)}$. What is the transform of the function which has value 1 in an elliptical region with major and minor semi-axes R_1 and R_2, and 0 outside this region?

6. Find the Fourier transform of the function $f(x)=x\exp(-x^2/2\sigma^2)$. What special relationship do you notice between the function and its transform?

7. Find the Fourier transform of a periodic saw-tooth wave defined in the period $(-\pi , \pi)$ as $f(x)=|x|$. How is the result related to that for a square wave?

8. The square wave f(x) with period 6b is shown in the figure below. Describe it using the convolution and other operations. Its Fourier transform can be written as a series of δ-functions: $F(k) = \sum_{m=-\infty}^{\infty} F_m \delta(k - m\beta)$. Find the values of β and F_m/F_1 for m=0, 2,3,4,5.



9. Show that the convolution of a Gaussian function with width σ_1 with a Gaussian of width σ_2 is a Gaussian of width $\sqrt{(\sigma_1^2 + \sigma_2^2)}$
(a) directly, by evaluating the convolution integral;
(b) by using the Fourier transform.

10. The auto-correlation function of f(x) is defined as $h(x) = \int_{-\infty}^{\infty} f(x')f^*(x - x')dx'$.

Show that its Fourier transform H(k) is real.

11. The error function e(x) is defined as the definite integral of a Gaussian:

$$e(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^{x} \exp(-\frac{t^2}{2\sigma^2})dt$$

Find its Fourier transform. Use the limit $\sigma \to 0$ to find the Fourier transform of the step function, f(x)=0 (for x<0), f(x)=1 (for x>0). Why do e(x) and f(x) not have formal Fourier transforms (although they are very useful in optics)?

has a Fourier transform similar in for $f(x) = \sum_{n=-\infty}^{\infty} \delta(x - na)$ 12. The comb function